Mathematics 595 — Advanced topics in category theory and higher category theory

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Course description:
Higher category theory is the study of structures which are like categories, but are “higher-dimensional”: while a category has objects (0 dimensions), and morphisms between objects (1 dimensions), higher dimensional analogues are allowed to have morphisms between morphisms (2 dimensions), and so on. Of special interest are “($\infty,1$)-categories”, in which higher morphisms are always invertible. These have become essential for new research in homotopy theory and related areas.

The overall goal of this course is to give an overview of the theory of ($\infty,1$)-categories, with an eye towards examples and applications coming from algebraic topology and allied fields. Here are some more specifics.

1. We’ll give a rapid overview of the basic theory of ($\infty,1$)-categories, using the model of quasicategories. The focus is on stating definitions and essential theorems rather than giving proofs (except in cases where it is especially illuminating to do so). There is a lot of work that needs to be done to get the theory off the ground, which can (and has) filled literally thousands of pages. Students who need to know more are directed to one of the many treatments now available, including: books by Joyal, Lurie, and Cisinski; kerodon.net, Lurie’s online textbook (in development); the textbook by Markus Land; and my own notes for previous 595 courses on higher categories.

2. In order for prepare to discuss a number of topics in $\infty$-categories, we prepare the way by presenting in detail analogous topics in the theory of 1-categories, in many cases using techniques borrowed from the $\infty$-categorical story. These include:
   - Straightening and unstraightening for 1-categories.
   - Presentable 1-categories, and the presentable tensor product.
   - Accessible 1-categories.
   - Monoidal structures for 1-categories.

3. Time permitting, we will outline the $\infty$-categorical analogues of these topics, along with other notions which don’t have a direct 1-categorical analogue, such as stable $\infty$-categories.
**Prerequisites:** Some familiarity with the basic notions of classical category theory is needed (e.g., functors, natural transformations, limits and colimits, etc.)

Familiarity with basic algebraic topology (e.g., fundamental group and singular homology, as in Math 525), or homological algebra, will be helpful, but not essential.

**Texts:** The main will be lecture notes, which will be available from my homepage. Other sources which take a similar point of view include: