GENERALIZING ACCESSIBLE ∞-CATEGORIES (DRAFT)

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Abstract. We describe a generalization of κ-filtered ∞-category, where the role regular cardinal κ is replaced by a class of small ∞-categories. This leads to a possible generalization of the notion of accessible ∞-category.

1. Introduction

This paper is a meditation on the notion of an *accessible ∞-category.* The underlying motivation is to find ways to talk about accessible ∞-categories which are model independent, and which in particular avoid as much as possible explicit reference to *regular cardinals.* In fact, we will introduce a generalization of regular cardinals, called a *regular class* of ∞-categories, and we will relate these to a (potential) generalization of the notion of accessible ∞-categories.

This program is heavily influenced by prior 1-categorical work, most notably the paper of Adamek, Borceux, Lack, and Rosicky [ABLR02], and in fact our results can be viewed as an ∞-categorical generalization of what they do However, what we will do here for ∞-categories seems to go beyond what has been done for 1-categories.

This is a DRAFT version of the paper, which I'm making available on my homepage. At this point, all the results I hope to include are stated here. Aside from general tidying up, the main thing that is missing are proofs for some ∞-categorical facts which seem to be clearly true, but whose proofs seem technically difficult, or are so far elusive for me. If you can provide any assistance with these matters, please let me know.

1.1. **Doctrines and filtered ∞-categories.** Given a regular cardinal κ, an ∞-category J is said to be *κ-filtered* if every map $f\colon K \to J$ from a κ-small simplicial set extends along the inclusion $K \subseteq K^{\triangleright}$ into the *right cone* of K. Recall that a simplicial set is *κ-small* it has fewer than κ non-degenerate simplices.

Here are two alternate characterizations of "κ-filtered".

1. J is *κ-filtered* if and only if the colimit functor $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves all κ-small limits [Lur09, 5.3.3.3].
2. J is *κ-filtered* if and only if the slice $J_{f/}$ is a weakly contractible simplicial set for every $f\colon K \to J$ from a κ-small simplicial set K [Lur09, 5.3.1.20–21].

Each of these admit a generalization, where we replace the class of "κ-small simplicial sets" with an arbitrary full subcategory $\mathcal{U}$ of the ∞-category Cat_∞ of small ∞-categories. In this case we make the following definitions.

1. An small ∞-category J is said to be *$\mathcal{U}$-filtered* if the colimit functor $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-limits for all $U \in \mathcal{U}$, where $\mathcal{S}$ is the ∞-category of (small) ∞-groupoids.
2. A small ∞-category J is said to be *weakly $\mathcal{U}$-filtered* if the slice $J_{f/}$ is a weakly contractible simplicial set for every functor $f\colon U^{\mathrm{op}} \to J$ for any $U \in \mathcal{U}$.

It turns out that (2) is equivalent to:

(2') The colimit functor $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-limits for all $U \in \mathcal{U}$ of diagrams of corepresentable functors (2.11).

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So $\mathcal{U}$-filtered implies weakly $\mathcal{U}$-filtered (2.13). We write $\mathrm{Filt}_{\mathcal{U}} \subseteq \mathrm{wFilt}_{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ for the classes of *small* $\mathcal{U}$-filtered and weakly $\mathcal{U}$-filtered ∞-categories.

We will be particularly interested the special case when the class $\mathcal{U}$ is a *doctrine*, by which we mean an essentially small full subcategory of Cat_∞.

Both of the above definitions recover Lurie's notion of κ-filtered ∞-category, where we take $\mathcal{U} := \kappa\text{-}\mathtt{sm}$, the "doctrine of κ-small ∞-categories" (5.1). They both also recover the notion of *sifted* ∞-category, where we take $\mathcal{U} := \mathtt{fin}\times$, the "doctrine of finite products" (3.14).

1.2. **Sound doctrines.** In general, there can be weakly $\mathcal{U}$-filtered ∞-categories which are not $\mathcal{U}$-filtered (6.3). We say that $\mathcal{U}$ is a *sound doctrine* if in fact $\mathrm{Filt}_{\mathcal{U}} = \mathrm{wFilt}_{\mathcal{U}}$.

There are a number of examples of sound doctrines. We have already mentioned several well-known examples: the doctrines $\kappa\text{-}\mathtt{sm}$ of κ-small ∞-categories for any regular cardinal κ, and the doctrine $\mathtt{fin}\times$ of finite products. Other examples are described in §3.

A sound doctrine of particular interest is $\mathtt{pb}$, the doctrine of pullbacks, whose only object is the walking cospan (4.1). We will say that a small ∞-category is *distilled* if it is $\mathtt{pb}$-filtered.

1.3. **Regular classes.** Every class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ has a "closure" $\overline{\mathcal{U}}$, defined so that $U \in \overline{\mathcal{U}}$ if and only if $\mathrm{colim}_J \colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-limits for all $J \in \mathrm{Filt}_{\mathcal{U}}$ (see §8).

We say that $\mathcal{K} \subseteq \mathrm{Cat}_\infty$ is a *regular class* if $\mathcal{K} = \overline{\mathcal{U}}$ for some *doctrine* $\mathcal{U}$. We refer to the corresponding class $\mathrm{Filt}_{\mathcal{K}}$ of small $\mathcal{K}$-filtered ∞-categories associated to a regular class as a *filtration class.* There is an evident bijection between the collections of regular classes and filtration classes.

We use the term "regular" to indicate that this is a kind of generalization of *regular cardinal.* In particular, $\kappa \mapsto \overline{\kappa\text{-}\mathtt{sm}}$ gives an embedding of the collection of regular cardinals into the collection of regular classes (8.1).

The filtration classes generalize the classical notion of ao *nullity class*, in sense that we have

$$\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} = \mathrm{Null}_{\mathcal{U}},$$

where the latter is the class of *$\mathcal{U}$-null ∞-groupoids*, i.e., the collection of ∞-groupoids X such that $X \to \mathrm{Fun}(U, X)$ is an equivalence for all $U \in \mathcal{U}$ (7.11). This indicates that a classification of regular classes is at least as difficult as a classification of nullity classes.

Both regular classes and filtration classes admit closure properties as full subcategories of Cat_∞: any regular class $\mathcal{K}$ is stable under $\mathcal{K}^{\mathrm{op}}$ colimits in Cat_∞, and any filtration class $\mathrm{Filt}_{\mathcal{K}}$ is closed under $\mathcal{K}$-filtered colimits in Cat_∞ (15.1).

1.4. **Subcategories of presheaves.** We can use a class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ of small ∞-categories to carve out various interesting subcategories of presheaf categories. For a small ∞-category C, we define (§9, §) the following full subcategories of presheaf category $\mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$:

- $\mathrm{Ind}_{\mathcal{U}}(C)$ consists of presheaves $X \in \mathrm{PSh}(C)$ such that the comma category (C/X) is $\mathcal{U}$-filtered.
- $\mathrm{wInd}_{\mathcal{U}}(C)$ consists of presheaves $X \colon C^{\mathrm{op}} \to \mathcal{S}$ such that the comma category (C/X) is weakly $\mathcal{U}$-filtered.
- $\mathrm{Flat}_{\mathcal{U}}(C)$ consists of presheaves $X \colon C^{\mathrm{op}} \to \mathcal{S}$ which are *$\mathcal{U}$-flat functors*, i.e., such that the left Kan extension $\widehat{X} := \mathrm{LKan}_\rho X$ along the Yoneda embedding $\rho \colon C \to \mathrm{PSh}(C)$, which is a functor $\widehat{X} \colon \mathrm{Fun}(C, \mathcal{S}) \to \mathcal{S}$, preserves U-limits for all $U \in \mathcal{U}$.

Under the additional hypothesis that C^{op} has all $\mathcal{U}$-limits, we also have the following (§13):

- $\mathrm{Lim}_{\mathcal{U}}(C)$ consists of presheaves $X \colon C^{\mathrm{op}} \to \mathcal{S}$ which preserve all $\mathcal{U}$-limits.

We will show (12.1), (12.3) that these subcategories, together with the subcategory of representable presheaves, form a chain of subcategories:

$$C \subseteq \mathrm{Ind}_{\mathcal{U}}(C) \subseteq \mathrm{Flat}_{\mathcal{U}}(C) \subseteq \mathrm{wInd}_{\mathcal{U}}(C) \subseteq \mathrm{PSh}(C).$$

Furthermore, if C^{op} has $\mathcal{U}$-limits, this extends to a chain (13.2), (13.4):

$$C \subseteq \mathrm{Ind}_{\mathcal{U}}(C) \subseteq \mathrm{Flat}_{\mathcal{U}}(C) \subseteq \mathrm{Lim}_{\mathcal{U}}(C) \subseteq \mathrm{wInd}_{\mathcal{U}}(C) \subseteq \mathrm{PSh}(C).$$

In particular, when $\mathcal{U}$ is sound, the chain collapses to a sequence of equalities:

$$\mathrm{Ind}_{\mathcal{U}}(C) = \mathrm{Flat}_{\mathcal{U}}(C) = \mathrm{wInd}_{\mathcal{U}}(C) \qquad (\text{and} = \mathrm{Lim}_{\mathcal{U}}(C) \text{ when defined}).$$

1.5. **Distilled ∞-categories.** As an illustration, we consider $\mathtt{pb} := \{\Lambda^2_2\}$, the set whose only element is the "walking span". This is the *doctrine of pullbacks*. Let J be a small ∞-category.

(1) Say J is *distilled* if it is weakly $\mathtt{pb}$-filtered, i.e., if and only if $J_{f/}$ is weakly contractible for every functor $f\colon \Lambda^2_0 \to J$.

By (2.11) this is formally equvialent to:

(2) J is distilled if and only if $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves pullbacks of spans of representable functors, i.e., of functors $\Lambda^2_2 \to J \xrightarrow{\rho} \mathrm{Fun}(J, \mathcal{S})$.

We show that $\mathtt{pb}$ is a sound doctrine (4.1). Therefore, the above definition is equivalent to:

(3) J is distilled if and only if $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves all pullbacks.

The regular class $\overline{\mathtt{pb}}$ generated by $\mathtt{pb}$ admits alterate characterizations. For instance (16.15):

(4) J is distilled if and only if $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves limits of all diagrams indexed by all weakly contractible finite simplicial sets, i.e., $\overline{\mathtt{pb}} = \overline{\mathcal{U}}$ where $\mathcal{U}$ is the class of weakly contractible ω-small ∞-categories.

We can also characterize distilled categories as a full subcategory of Cat_∞ (16.9):

(5) J is distilled if and only if it is equivalent to the colimit of a functor $f\colon X \to \mathrm{Cat}_\infty$, where X is a small ∞-groupoid and f takes values in ω-filtered ∞-categories.

As a consequence, we learn that (an ∞-category has distilled colimits/a functor preserves distilled colimits) if and only if it (has/preserves) (i) ω-filtered colimits and (ii) colimits indexed by ∞-groupoids (16.11).

1.6. **Generalized accessibility.** Of these subcategories, the one we have labelled $\mathrm{Ind}_{\mathcal{U}}(C)$ bears some striking similarities to the categories $\mathrm{Ind}_\kappa(C)$ defined by Lurie. In particular:

- $\mathrm{Ind}_{\mathcal{U}}(C)$ is the *free $\mathcal{U}$-filtered colimit completion* of C, so that $\mathrm{Ind}_{\mathcal{U}}(C)$ has $\mathcal{U}$-filtered colimits (10.1), and every functor $f\colon C \to A$ to a category with $\mathcal{U}$-filtered colimits extends essentially uniquely to a $\mathcal{U}$-filtered colimit preserving functor $\widehat{f}\colon \mathrm{Ind}_{\mathcal{U}}(C) \to A$ (10.3).
- If in addition f is fully faithful and takes values in the full subcategory of *$\mathcal{U}$-compact* objects of A, then $\widehat{f}$ is also fully faithful (11.3).
- Thus, if furthermore A is generated under $\mathcal{U}$-filtered colimits by the essential image of f, then $\mathrm{Ind}_{\mathcal{U}}(C) \xrightarrow{\sim} A$.

For instance, if $\mathtt{fin}\times$ is the *doctrine of finite products*, then $\mathrm{Ind}_{\mathtt{fin}\times}(C)$ is the *free sifted colimit completion* of C. In the case that C has finite coproducts, $\mathrm{Ind}_{\mathtt{fin}\times}(C) = \mathrm{Lim}_{\mathtt{fin}\times}(C)$ is an example of a *non-abelian derived category* as in [Lur09, 5.5.8].

We will say that an ∞-category is *$\mathcal{U}$-accessible* if it is equivalent to $\mathrm{Ind}_{\mathcal{U}}(C)$ for some small ∞-category C and doctrine $\mathcal{U}$, and *generalized accessible* if it is $\mathcal{U}$-accessible for some doctrine $\mathcal{U}$.

This last definition begs a question: are there any generalized accessible ∞-categories which are not accessible in Lurie's sense? We have left this as unresolved. What we can show is that $\mathrm{Flat}_{\mathcal{U}}(C)$ is accessible (14.1), which implies that $\mathcal{U}$-accesibility implies accessibility for all sound doctrines $\mathcal{U}$.

1.7. **Comparison to 1-categorical work.** This work is partly inspired by the paper [ABLR02], from which I have adapted several notions (e.g., $\mathcal{U}$-filtered categories, sound doctrines, etc.) and some of the results in this paper are simply ∞-categorical upgrades of theirs.

There are some key differences in approach. In [ABLR02], the authors define "$\mathcal{U}$-accessible categories" to be categories of *$\mathcal{U}$-flat functors* from a small category to sets, analogous to our ∞-category $\mathrm{Flat}_{\mathcal{U}}(C)$ of $\mathcal{U}$-flat functors $C \to \mathcal{S}$. Unfortunately, the category of $\mathcal{U}$-flat functors does not seem to come with a convenient universal property, eg., it is not in general the same as the $\mathcal{U}$-filtered colimit completion of C, so to enforce this these authors must assume an additional condition on $\mathcal{U}$. Thus, the notion of "sound doctrine" plays a necessary role in their approach.

In our point of view, the correct notion of "$\mathcal{U}$-accessible category" corresponds to what we have call $\mathrm{Ind}_{\mathcal{U}}(C)$, as this has the desired universal property in all cases, without the "soundness" restriction. So for us the key notion becomes that of *regular class*, which precisely classify the "types" of $\mathcal{U}$-accessibility. Thus a crucial question (as yet unresolved) becomes: is every regular class generated by some sound doctrine? An affirmative answer would show that generalized accessibility is the same as ordinary accessibility.

We remark that a correspondence between regular classes and filtration classes (as part of a "Galois connection" on the collection of classes of categories), is briefly discussed in [BJLS15], as are some results related to results on $\mathcal{U}$-filtered ∞-groupoids §7.

There are other differences in detail, which mainly come down to the fact that [ABLR02] consider functors to *sets*, while we consider functors to *∞-groupoids*. For instance:

- Where we have a condition that an ∞-category be *weakly contractible*, they typically only require that a 1-category be *connected*. This distinction appears in the descriptions of *terminally filtered* 1-categories or ∞-categories (3.8), (3.9). It also appears in the definition of (what we have called) *weakly $\mathcal{U}$-filtered* (2.8).
- The doctrine of pullbacks is not sound in the 1-categorical context, but is sound in our setting (4.4).

1.8. **Acknowledgements.** Since I'm not well-versed in modern ∞-categorical techniques, I have benefited greatly from assistence provided by various people in response to my questions (very often in online forums such as the Algebraic topology Discord server, and the Homotopy Theory chatroom on MathOverflow). Those who I would like to thank include:

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- Clark Barwick, Rune Haugseng: an isofibration $D \to G$ with $G \in \mathcal{S}$ presents D as a colimit of $f\colon G \to \mathrm{Cat}_\infty$.
- Tim Campion, Shachar Carmeli, Denis-Charles Cisinski, Piotr Pstragowksi: accessibility of $\mathrm{Flat}_{\mathcal{U}}(C)$.
- Classifying slices (2.10): Maxime Ramzi and Dylan Wilson.[1]

1.9. **Standard notation.** I'm basically using Lurie's notation (mostly).

We assume a fixed universe. We write Cat_∞ for the ∞-category of small ∞-categories, and $\mathcal{S} \subseteq \mathrm{Cat}_\infty$ for the full subcategory of ∞-groupoids.

I write $\mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ for the presheaf category, and $\rho\colon C \to \mathrm{PSh}(C)$ for the Yoneda functor.

[1]See answers to "What functors are classified by slices of ∞-categories?" `https://mathoverflow.net/q/381549`.

2. Notions of filtered ∞-categories

In this section, we describe several notions of "filtered" ∞-categories, based not on regular cardinals, but rather on arbitrary classes of ∞-categories. Both are based on concepts introduced in a 1-categorical context in [ABLR02].

2.1. **Classes and doctrines.** In this paper, we will often speak of a **class** $\mathcal{U}$ of small ∞-categories. Given such a class, we use the same notation $\mathcal{U}$ for the full subcategory of Cat_∞ spanned by $\mathcal{U}$.

We say that $\mathcal{U}$ is a **doctrine** if it is essentially small, i.e., if there exists a set S of small ∞-categories such that every object in $\mathcal{U}$ is equivalent to one in S.

We say that a functor $f\colon C \to D$ between ∞-categories **preserves $\mathcal{U}$-limits** if f preserves any U-limits which exist in C, for all $U \in \mathcal{U}$. We also use the analogous term **preserves $\mathcal{U}$-colimits**.

2.2. *Remark.* Another term for doctrine is **limit doctrine**, because we will typically be concerned with limits of functors from elements of a doctrine. Thus it is often convenient describe a doctrine by describing instead its limits. For instance, the class $\mathtt{pb} = \{\Lambda^2_2\}$ which contains only the walking cospan can be referred to as the "doctrine of pullbacks".

2.3. **$\mathcal{U}$-filtered ∞-categories.** We say that $J \in \mathrm{Cat}_\infty$ is **$\mathcal{U}$-filtered** if the colimit functor

$$\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$$

preserves $\mathcal{U}$-limits.

2.4. *Remark.* Here is an equivalent formulation of $\mathcal{U}$-filtered. Let $\mathrm{Fun}^{\lim}(U^{\triangleleft}, \mathcal{S}) \subseteq \mathrm{Fun}(U^{\triangleleft}, \mathcal{S})$ denote the full subcategory of limit cones. Then J is $\mathcal{U}$-filtered exactly if for every $U \in \mathcal{U}$, the full subcategory $\mathrm{Fun}^{\lim}(U^{\triangleleft}, \mathcal{S})$ is stable under J-colimits in $\mathrm{Fun}(U^{\triangleleft}, \mathcal{S})$. **(This needs justification.)**

We write $\mathrm{Filt}_{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ for the class of small $\mathcal{U}$-filtered ∞-categories. Note that $\mathcal{U} \subseteq \mathcal{V}$ implies $\mathrm{Filt}_{\mathcal{U}} \supseteq \mathrm{Filt}_{\mathcal{V}}$, and also that $\mathrm{Filt}_{\bigcup \mathcal{U}_i} = \bigcap \mathrm{Filt}_{\mathcal{U}_i}$.

2.5. *Remark.* Our definition of "$\mathcal{U}$-filtered" imitates that of [ABLR02, 1.2] for 1-categories, where the role of ∞-groupoids is replaced with sets. We warn that the 1-categorical and ∞-categorical notions are distinct, even when we restrict to 1-categories, ultimately because of the role of sets vs. ∞-groupoids.

For instance, if $\mathtt{term} = \{\varnothing\}$ (the "doctrine of the terminal object"), then any connected but non-contractible 1-category is $\mathtt{term}$-filtered in the sense of [ABLR02] (see [ABLR02, 1.3(vii)]), but not in our sense (3.8).

Likewise, if $\mathtt{pb} = \{\Lambda^2_2\}$ (the "doctrine of pullbacks"), then any non-trivial groupoid is $\mathtt{pb}$-filtered in our sense (as a consequence of (7.11)), but not in the sense of [ABLR02] (see [ABLR02, 2.3(vii)]).

2.6. **Weakly $\mathcal{U}$-filtered ∞-categories.** We say that $J \in \mathrm{Cat}_\infty$ is **weakly $\mathcal{U}$-filtered** if if for every $U \in \mathcal{U}$ and every functor $f\colon U^{\mathrm{op}} \to J$, the slice category $J_{f/}$ is weakly contractible as a simplicial set.

We write $\mathrm{wFilt}_{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ for the class of small weakly $\mathcal{U}$-filtered ∞-categories. Note that $\mathcal{U} \subseteq \mathcal{V}$ implies $\mathrm{wFilt}_{\mathcal{U}} \supseteq \mathrm{wFilt}_{\mathcal{V}}$, and also that $\mathrm{wFilt}_{\bigcup \mathcal{U}_i} = \bigcap \mathrm{wFilt}_{\mathcal{U}_i}$.

2.7. **Proposition.** *If J^{op} has all $\mathcal{U}$-limits then J is weakly $\mathcal{U}$-filtered.*

Proof. This hypothesis implies that J has U^{op}-colimits for all $U \in \mathcal{U}$, so for every $f\colon U^{\mathrm{op}} \to J$ the slice $J_{f/}$ has an initial object and so is contractible. □

2.8. *Remark.* Our definition of "weakly $\mathcal{U}$-filtered" is also inspired by [ABLR02], where the 1-categorical analog plays an important role (but is not given a name). However, there is one significant difference: in [ABLR02], the analogous definition merely requires that the slices $J_{f/}$ be *connected*, rather than contractible (e.g., as in [ABLR02, 2.1 or 2.2], where this slice is identical to what they call the "category of cocones").

2.9. **$\mathcal{U}$-filtered implies weakly $\mathcal{U}$-filtered.** We begin by giving an alternate characterization of weakly $\mathcal{U}$-filtered ∞-categories.

2.10. **Lemma.** *For any functor $f\colon C \to D$ between small ∞-categories, the evident forgetful functor $\pi\colon D_{f/} \to D$ is a left fibration classified by*

$$F := \lim_{C^{\mathrm{op}}}(\rho \circ f^{\mathrm{op}})\colon D \to \mathcal{S},$$

where $\rho\colon C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathcal{S})$ denotes the Yoneda functor.

Proof. (Thanks to Dylan Wilson for this proof.) There exists a homotopy pullback square of quasicategories of the form

$$\begin{tikzcd} D_{f/} \arrow[r] \arrow[d, "\pi"'] & \mathrm{Fun}(C,D)_{(f)/} \arrow[d, "\pi'"] \\ D \arrow[r, "\gamma"'] & \mathrm{Fun}(C,D) \end{tikzcd}$$

where π and π' are the evident forgetful functors (which are left fibrations), and γ is adjoint to $C \to \{\mathrm{id}_D\} \to \mathrm{Fun}(D, D)$. Here "$\mathrm{Fun}(C, D)_{(f)/}$" denotes the slice over the object f of $\mathrm{Fun}(C, D)$. To see this is a homotopy pullback, observe that the analogous square in which the slices are *alternate slices* [Lur09, 4.2.1] is a pullback of simplicial sets.

We know that π' is classified by the corepresentable functor $\mathrm{Map}_{\mathrm{Fun}(C,D)}(f, -)\colon \mathrm{Fun}(C, D) \to \mathcal{S}$, and therefore π is classified by $\mathrm{Map}_{\mathrm{Fun}(C,D)}(f, \gamma(-)) \approx \lim_{C^{\mathrm{op}}} \mathrm{Map}_D(f, -)$. □

2.11. **Proposition.** *A small ∞-category J is weakly $\mathcal{U}$-filtered if and only if $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-limits of corepresentables for all $U \in \mathcal{U}$.*

Proof. For any functor $f\colon U \to J^{\mathrm{op}}$, we have (2.10) that the left fibration $J_{f^{\mathrm{op}}/} \to J$ is classified by $F\colon J \to \mathcal{S}$ with $F \approx \lim_U(\rho \circ f)$ where $\rho\colon J^{\mathrm{op}} \to \mathrm{Fun}(J^{\mathrm{op}}, \mathcal{S})$ is the Yoneda functor. Therefore $\mathrm{colim}_J F \approx \mathrm{colim}_J \lim_U(\rho \circ f)$ is weakly equivalent to the simplicial set $J_{f^{\mathrm{op}}/}$.

On the other hand, for any object $j \in J$, we have that $\mathrm{colim}_J \mathrm{Map}_J(j, -)$ is a terminal object of $\mathcal{S}$, and thus $\lim_U \mathrm{colim}_J(\rho \circ f)$ is also a terminal object.

Thus, if J is weakly $\mathcal{U}$-filtered, the tautological map $\tau\colon\ \mathrm{colim}_J \lim_U(\rho \circ f) \to \lim_U \mathrm{colim}_J(\rho \circ f)$ is a weak equivalence, since both source and target of τ are contractible.

Conversely, $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-limits of corepresentables exactly if τ is always a weak equivalence, and thus all $J_{f^{\mathrm{op}}/} \approx \mathrm{colim}_J \lim_U(\rho \circ f)$ are contractible, whence J is weakly $\mathcal{U}$-filtered. □

2.12. *Remark.* In the 1-categorical context, the analog of this is also true, where "weakly $\mathcal{U}$-filtered" is taken to mean "slices $J_{f/}$ are connected", and we replace the role of $\mathcal{S}$ with the category of sets.

2.13. **Corollary.** *Every $\mathcal{U}$-filtered ∞-category is weakly $\mathcal{U}$-filtered:* $\mathrm{Filt}_{\mathcal{U}} \subseteq \mathrm{wFilt}_{\mathcal{U}}$.

2.14. *Remark.* Combining (2.7) and (2.11), we see that if J^{op} has $\mathcal{U}$-limits, then $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves $\mathcal{U}$-limits of corepresentables. It is very easy to see this implication directly, since the Yoneda functor $\rho\colon J^{\mathrm{op}} \to \mathrm{Fun}(J, \mathcal{S})$ preserves all limits which exist in J^{op}, while the colimit functor colim_J takes any corepresentable functor to a terminal object in $\mathcal{S}$, and of course any limit of terminal objects in $\mathcal{S}$ is a terminal object.

2.15. **Closure properties of $\mathrm{Filt}_{\mathcal{U}}$ and $\mathrm{wFilt}_{\mathcal{U}}$.**

2.16. **Proposition.** *If C and D are equivalent ∞-categories, then C is $\mathcal{U}$-filtered if and only if D is, and C is weakly $\mathcal{U}$-filtered if and only if D is.*

Proof. . . . □

We note the following inheritance property for $\mathrm{Filt}_{\mathcal{U}}$, involving the notion of a *cofinal functor* [Lur09, 4.1.1].

2.17. **Proposition.** *Consider a class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$. If $v\colon J \to K$ a cofinal functor of small ∞-categories such that J is $\mathcal{U}$-filtered, then K is also $\mathcal{U}$-filtered.*

Proof. Since v is cofinal, restriction along the induced functor $v^\triangleright\colon J^\triangleright \to K^\triangleright$ between right cones takes K-colimit cones to J-colimit cones [Lur09, 4.1.1.8]. Therefore the diagram

$$\begin{tikzcd} \mathrm{Fun}(K,\mathcal{S}) \arrow[r, "v^*"] \arrow[dr, "\mathrm{colim}_K"'] & \mathrm{Fun}(J,\mathcal{S}) \arrow[d, "\mathrm{colim}_J"] \\ & \mathcal{S} \end{tikzcd}$$

commutes in the homotopy category of ∞-categories. Since the restriction functor v^* preserves all limits, the claim follows.

. . . □

2.18. *Remark.* It is not the case that if J is weakly $\mathcal{U}$-filtered and $J \to K$ a cofinal functor, then K is weakly $\mathcal{U}$-filtered: see (7.4).

Both $\mathrm{Filt}_{\mathcal{U}}$ and $\mathrm{wFilt}_{\mathcal{U}}$ are stable under finite products in Cat_∞.

2.19. **Proposition.** *The terminal ∞-category is $\mathcal{U}$-filtered. If J, K are $\mathcal{U}$-filtered, then $J \times K$ is $\mathcal{U}$-filtered.*

Proof. That the terminal ∞-category is $\mathcal{U}$-filtered is immediate.

We can factor the colimit functor $\mathrm{colim}_{J\times K}\colon \mathrm{Fun}(J\times K, \mathcal{S}) \to \mathcal{S}$ as a composite

$$\mathrm{Fun}(J, \mathrm{Fun}(K,\mathcal{S})) \xrightarrow{\mathrm{Fun}(J,\mathrm{colim}_K)} \mathrm{Fun}(J,\mathcal{S}) \xrightarrow{\mathrm{colim}_J} \mathcal{S}.$$

Note that if $F\colon A \to B$ is any functor between ∞-categories with $\mathcal{U}$-limits which preserves $\mathcal{U}$-limits, then the induced functor $\mathrm{Fun}(C,F)\colon \mathrm{Fun}(C,A) \to \mathrm{Fun}(C,B)$ also preserved $\mathcal{U}$-limits, because limits in functor categories can be computed object-by-object. Thus the composite functor preserves $\mathcal{U}$-limits, so $J \times K$ is $\mathcal{U}$-filtered. □

2.20. **Proposition.** *The terminal ∞-category is weakly $\mathcal{U}$-filtered. If J, K are weakly $\mathcal{U}$-filtered, then $J \times K$ is weakly $\mathcal{U}$-filtered.*

Proof. Immediate from the isomorphisms $\Delta^0_{f/} \approx \Delta^0$ and $(J \times K)_{(f,g)/} \approx J_{f/} \times K_{g/}$. □

Both $\mathrm{Filt}_{\mathcal{U}}$ and $\mathrm{wFilt}_{\mathcal{U}}$ are stable under retracts.

2.21. **Proposition.** *Suppose $s\colon C \to D$ and $r\colon D \to C$ are functors such that rs is naturally isomorphic to the identity functor. Then if C is $\mathcal{U}$-filtered so is D, and if C is weakly $\mathcal{U}$-filtered so is D.*

Proof. Because these properties are equivalence invariant (2.16), without loss of generality we can assume $rs = \mathrm{id}_C$.

. . .

The statement for weak $\mathcal{U}$-filtered is immediate from functoriality of slice: given $f\colon U^{\mathrm{op}} \to C$, we see that $C_{f/}$ is a retract of $D_{/fs}$. □

2.22. **Sound classes and doctrines.** We say that a class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ is a **sound class** if all small weakly $\mathcal{U}$-filtered ∞-categories are $\mathcal{U}$-filtered, i.e., if $\mathrm{Filt}_{\mathcal{U}} = \mathrm{wFilt}_{\mathcal{U}}$. We call a class $\mathcal{U}$ a **sound doctrine** if it is a doctrine and a sound class. (The "sound doctrine" terminology is taken from [ABLR02, 2.1].)

As we will see, there are many examples of both sound doctrines and unsound doctrines.

3. Examples of $\mathcal{U}$-filtered ∞-categories

We describe a number of "standard" examples of classes $\mathcal{U}$ and associated classes $\mathrm{Filt}_{\mathcal{U}}$ of $\mathcal{U}$-filtered ∞-categories. All of the exmaples of $\mathcal{U}$ in this section will be sound, so in each case we are also describing $\mathrm{wFilt}_{\mathcal{U}}$.

3.1. **Universally filtered.** We consider $\mathtt{univ} := \mathrm{Cat}_\infty$, the class of all small ∞-categories. Thus write $\mathrm{Filt}_{\mathtt{univ}}$ and $\mathrm{wFilt}_{\mathtt{univ}}$ are the corresponding classes of Cat_∞-filtered and weakly Cat_∞-filtered ∞-categories. As we see below, these two classes are the same: we will call this the class of **universally filtered** ∞-categories. Note that universally filtered ∞-categories are $\mathcal{U}$-filtered (and weakly $\mathcal{U}$-filtered) for *every* class $\mathcal{U}$.

3.2. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$. The following are equivalent.*

1. *$J \in \mathrm{Filt}_{\mathtt{univ}}$.*
2. *$J \in \mathrm{wFilt}_{\mathtt{univ}}$.*
3. *The evident inclusion $J \rightarrowtail J^{\triangleright}$ admits a retraction.*
4. *J is κ-filtered for every regular cardinal κ.*
5. *The Karoubi completion (i.e., idempotent completion) J^+ of J has a terminal object.*

Proof.
(1) ⇒ (2) is by (2.13).

(2) ⇒ (3): If J is weakly universally filtered, then the slice $J_{\mathrm{id}/}$ of the identity functor $\mathrm{id}\colon J \to J$ is weakly contractible, and therefore non-empty, and any object of this slice corresponds exactly to a choice of retraction of $J \rightarrowtail J^{\triangleright}$.

(3) ⇒ (4) is [Lur09, 5.3.1.9].

(4) ⇒ (1): [Lur09, 5.3.3.3] says that κ-filtered colimits preserve κ-small limits, so if J is κ-filtered for all κ, then J-colimits preserve all small limits.

(5) ⇒ (1): Note that if J itself has a terminal object t, then $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$. is equivalent to evaluation at t, and so preserves all limits. For the general statement note that the evident restriction functor $i^*\colon \mathrm{Fun}(J^+, \mathcal{S}) \to \mathrm{Fun}(J, \mathcal{S})$ is an equivalence and is compatible with taking J or J^+ colimits.

(1) ⇒ (5): By hypothesis the functor $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves both limits and colimits. In particular the functor is accessible, so it is corepresented by an object $A \in \mathrm{Fun}(J, \mathcal{S})$ [Lur09, 5.5.2.7]. Since colim_J preserves small colimits the object A is "completely compact" in the sense of [Lur09, 5.1.6.5], and so $A \in \mathrm{Fun}(J, \mathcal{S})$ is a retract of some corepresentable functor in $\mathrm{Fun}(J, \mathcal{S})$. Thus, by the proof of [Lur09, 5.1.4.2], A can be identified as on object of the idempotent completion $(J^+)^{\mathrm{op}}$ of J^{op}, and in fact it is an initial object of this, since $\mathrm{Map}(A, \rho(j)) \approx \mathrm{colim}_J\, \rho(j) \approx *$ for any object j in J. Therefore A corresponds to a terminal object of J^+ as desired. □

3.3. *Example.* The "walking idempotent" Idem is universally filtered [Lur09, 5.3.1.9].

3.4. **Trivially filtered.** We consider $\mathcal{U} = \varnothing$, the empty class of ∞-categories (the "trivial doctrine"). The following is immediate.

3.5. **Proposition.** $\mathrm{Filt}_\varnothing = \mathrm{wFilt}_\varnothing = \mathrm{Cat}_\infty$. *Thus, the trivial doctrine is sound.*

We can also consider the variant $\mathcal{U} = \{\Delta^0\}$, the class consisting only of the terminal ∞-category.

3.6. **Proposition.** $\mathrm{Filt}_{\{\Delta^0\}} = \mathrm{wFilt}_{\{\Delta^0\}} = \mathrm{Cat}_\infty$.

Proof. $\mathrm{colim}\colon \mathrm{Fun}(\Delta^0, \mathcal{S}) \to \mathcal{S}$ is an equivalence, so preserves all limits, while for any $f\colon (\Delta^0)^{\mathrm{op}} \to J$ the slice $J_{f/}$ is weakly contractible. □

3.7. **Terminally filtered.** We consider $\mathtt{term} := \{\varnothing\}$, the class consisting only of the initial ∞-category. This is the "doctrine of the terminal object" We write $\mathrm{Filt}_{\mathtt{term}}$ and $\mathrm{wFilt}_{\mathtt{term}}$ for the corresponding classes of infinity categories. As we see below, these two classes are the same: we will call this the class of **terminally filtered** ∞-categories.

3.8. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$. The following are equivalent.*

(1) *$J \in \mathrm{Filt}_{\mathtt{term}}$.*
(2) *$J \in \mathrm{wFilt}_{\mathtt{term}}$.*
(3) *J is weakly contractible.*

Thus, the doctrine of the terminal object is sound.

Proof.
(1) $\Rightarrow$ (2) is by (2.13).
(2) $\Rightarrow$ (3) is immediate since $J_{f/} = J$ when $f\colon \varnothing^{\mathrm{op}} \to J$.
(3) $\Rightarrow$ (1) since for $f\colon \varnothing \to \mathcal{S}$ we have that $\mathrm{colim}_J \lim_\varnothing f \approx \mathrm{colim}_J *$ is weakly equivalent to J, while $\lim_\varnothing \mathrm{colim}_J f \approx \lim_\varnothing \varnothing$ is contractible. ∎

3.9. *Remark.* In the 1-categorical context in which one only considers presheaves of sets, we have that colim_J preserves U-limits of sets iff U is connected.

3.10. **Binary-product-filtered ∞-categories.** We consider $\times := \{\partial\Delta^1\}$, the class consisting only of the discrete ∞-groupoid with 2-objects. This is the "doctrine of binary products". We write $\mathrm{Filt}_\times$ and $\mathrm{wFilt}_\times$ for the corresponding classes of infinity categories.

3.11. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$. The following are equivalent.*

(1) *$J \in \mathrm{Filt}_\times$.*
(2) *$J \in \mathrm{wFilt}_\times$.*
(3) *The diagonal functor $\delta\colon J \to J \times J$ is cofinal.*

Thus, the doctrine of binary products is sound.

Proof.
(1) $\Rightarrow$ (2) is by (2.13).
(2) $\Leftrightarrow$ (3): Given a pair of objects j_1, j_2 in J, let $f\colon \partial\Delta^1 \to J$ be the functor determined by this pair. An elementary argument shows that there is a pullback of simplicial sets

$$\begin{tikzcd} J_{f/} \arrow[r] \arrow[d] & (J \times J)_{(j_1,j_2)/} \arrow[d] \\ J \arrow[r, "\delta"'] & J \times J \end{tikzcd}$$

where the vertical maps are the evident forgetful functors. By [Lur09, 4.1.3.1], δ is cofinal exactly if the pullback $J_{f/}$ is weakly contractible for all objects (j_1, j_2) of $J \times J$.
(3) $\Rightarrow$ (1). This follows exactly as in the proof of [Lur09, 5.5.8.11], which for preservation of binary products relies only on [Lur09, 5.5.8.6], which in turn only relies on the hypothesis that δ is cofinal. ∎

3.12. *Remark.* The empty ∞-category $\varnothing$ is in $\mathrm{Filt}_\times$. As we see below (§3.13), it is the only object of $\mathrm{Filt}_\times$ which is not sifted, so $\mathrm{Filt}_\times = \mathrm{Filt}_{\mathtt{fin}\times} \cup \{\varnothing\}$.

Furthermore, we have that $\mathrm{Filt}_\times = \mathrm{Filt}_{\{1,\dots,n\}}$ for any finite $n \geq 2$. To see this, note that evidently $\mathrm{Filt}_\times \subseteq \mathrm{Filt}_{\{1,\dots,n\}}$, while if $J \in \mathrm{Filt}_{\{1,\dots,n\}}$ and $J \neq \varnothing$ then J is weakly contractible (see the proof of [Lur09, 5.5.8.7]), and thus J is sifted.

3.13. **Sifted ∞-categories.** We consider the class $\mathtt{fin}\times$ of finite discrete simplicial sets. This is the "doctrine of finite products". We write $\mathrm{Filt}_{\mathtt{fin}\times}$ and $\mathrm{wFilt}_{\mathtt{fin}\times}$ of the corresponding classes.

Recall that an ∞-category J is **sifted** when (i) J is non-empty and (ii) the diagonal $\delta\colon J \to J \times J$ is cofinal [Lur09, 5.5.8.1]. Also note that we can replace condition (i) with condition (i') J is weakly contractible [Lur09, 5.5.8.7].

3.14. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$. The following are equivalent.*

(1) $J \in \mathrm{Filt}_{\mathtt{fin}\times}$.
(2) $J \in \mathrm{wFilt}_{\mathtt{fin}\times}$.
(3) *J is sifted.*

Thus, the doctrine of finite products is sound.

Proof. Let $\mathcal{U} = \{\varnothing, \partial\Delta^1\}$. Then it is immediate from (3.8) and (3.11) that $\mathrm{Filt}_{\mathcal{U}} = \mathrm{wFilt}_{\mathcal{U}}$ is the class of small sifted ∞-categories.

It is clear that $\mathrm{Filt}_{\mathcal{U}} = \mathrm{Filt}_{\mathtt{fin}\times}$, since a functor preserves finite products if and only if it preserves binary products and the terminal object.

...

□

3.15. *Remark.* It is easy to see that $\mathrm{Filt}_{\mathtt{fin}\times} = \mathrm{Filt}_{\{\varnothing,\partial\Delta^1\}}$.

4. Distilled ∞-categories

We consider $\mathtt{pb} := \{\Lambda^2_2\}$, the class consisting only of the walking cospan. This is the "doctrine of pullbacks". We write $\mathrm{Filt}_{\mathtt{pb}}$ and $\mathrm{wFilt}_{\mathtt{pb}}$ for the corresponding classes of ∞-categories.

4.1. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$. The following are equivalent.*

(1) $J \in \mathrm{Filt}_{\mathtt{pb}}$.
(2) $J \in \mathrm{wFilt}_{\mathtt{pb}}$.

Thus, the doctrine of pullbacks is sound.

We will call such objects **distilled** ∞-categories (thus[2] perpetuating the analogy with "filtered" and "sifted").

Proof. (1) $\Rightarrow$ (2) by (2.13).

(2) $\Rightarrow$ (1) is a consequence of (2.11), together with a special case of the following proposition about ∞-topoi (4.2), with $\mathcal{X} = \mathcal{S}$ and $J = C^{\mathrm{op}}$, and $f = \mathrm{colim}_J$. □

4.2. **Proposition.** *Let $C \in \mathcal{S}$, let $\mathcal{X}$ be an ∞-topos, and let $f\colon \mathrm{PSh}(C) \to \mathcal{X}$ be a colimit preserving functor. If f preserves pullback squares in $\mathrm{PSh}(C)$ whose underlying cospan is a diagram of representable presheaves, then f preserves all pullbacks.*

Proof. The proof is contained in the proof of [Lur09, 6.1.5.2]; the requirement there that C have pullbacks is not actually needed. I briefly sketch a proof here.

We will use the "descent principle" for ∞-topoi (applied to $\mathcal{X}$, and to $\mathrm{PSh}(C)$ which is also an ∞-topos). Let $\mathrm{Cart}(\mathcal{X}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{X})$ denote the (non-full) subcategory with all objects, whose morphisms are the *Cartesian maps*, i.e., the edges $\alpha\colon p \to q$ in $\mathrm{Fun}(\Delta^1, \mathcal{X})$ such that the square

$$\begin{tikzcd} p(0) \arrow[r] \arrow[d, "\alpha(0)"] & q(0) \arrow[d] \\ p(1) \arrow[r, "\alpha(1)"'] & q(1) \end{tikzcd}$$

[2] regretably?

is a pullback in $\mathcal{X}$. The descent principle says that (i) $\mathrm{Cart}(\mathcal{X})$ has all small colimits and (ii) $\mathrm{Cart}(\mathcal{X}) \to \mathrm{Fun}(\Delta^1, \mathcal{X})$ preserves small colimits. (This property is (1) $\Rightarrow$ (2) of [Lur09, 6.1.0.6]. There it is stated it terms of the notion of a *Cartesian transformation* $\alpha\colon p \to q$ of functors $K \to \mathcal{X}$, which is the same thing as a functor $\alpha'\colon K \to \mathrm{Cart}(\mathcal{X})$.)

We can also consider $\mathrm{Cart}^2(\mathcal{X}) := \mathrm{Cart}(\mathrm{Cart}(\mathcal{X})) \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{X})$. This is a (non-full) subcategory whose objects are pullback squares, and whose morphisms are cubes such that each face is a pullback. The above descent principle then implies that (i) $\mathrm{Cart}^2(\mathcal{X})$ has all small colimits and (ii) $\mathrm{Cart}^2(\mathcal{X}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{X})$ preserves small colimits.

Say that a morphism $\alpha\colon Y \to Z$ in $\mathrm{PSh}(C)$ is *good* if f takes every pullback square in $\mathrm{PSh}(C)$ of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ X \arrow[r] & Z \end{tikzcd} \tag{4.3}$$

to a pullback square in $\mathcal{X}$. Say that an object Z of $\mathrm{PSh}(C)$ is *good* if every morphism $\alpha\colon Y \to Z$ with target Z is good. We proceed via the following steps.

(1) Show that every morphism $\alpha\colon \rho(y) \to \rho(z)$ between representable presheaves is good.
(2) Show that every representable presheaf $\rho(z)$ is good.
(3) Show that every object Z of $\mathrm{PSh}(C)$ is good, and so the proposition is proved.

Note that every object of $\mathrm{Cart}(\mathcal{X})$ (i.e., morphism $P \to X$ in $\mathrm{PSh}(C)$) is a small colimit of a functor $I \to \mathrm{Cart}(\mathcal{X})$ such that each $i \in I$ is sent to an object $P_i \to X_i$ where X_i is a representable presheaf. Then (1) follows by resolving $P \to X$ as a colimit of of objects whose target is representable, and using the hypothesis of the proposition together with the descent principles for $\mathrm{PSh}(C)$ and $\mathcal{X}$. Then (2) follows similarly, by resolving $P \to Y$ as a colimit of objects whose target is representable, and using (1) and the descent principles for $\mathrm{PSh}(C)$ and $\mathcal{X}$.

Next note that every object of $\mathrm{Cart}^2(\mathcal{X})$ (i.e., pullback square in $\mathrm{PSh}(C)$) is a small colimit of a functor $I \to \mathrm{Cart}^2(\mathcal{X})$ such that each $i \in I$ is sent to a pullback square where the lower right corner is a representable presheaf. Then (3) follows by resolving the square in this way, using (2) and the descent principles for $\mathrm{PSh}(C)$ and $\mathcal{X}$. $\square$

4.4. *Remark.* In the 1-categorical context, the doctrine of pullbacks is *not sound* [ABLR02, 2.3(vii)]. Ultimately, this is because the inclusion $\mathrm{Set} \subseteq \mathcal{S}$ does not preserve ∞-categorical colimits.

4.5. *Example.* Recall that a functor $f\colon A \to B$ from an A which has finite limits preserves finite limits if and only if it preserves terminal objects and pullbacks [Lur09, 4.4.2.5]. Therefore

$$\mathrm{Filt}_{\{\varnothing, \Lambda^2_2\}} = \mathrm{Filt}_{\mathtt{term}} \cap \mathrm{Filt}_{\mathtt{pb}}$$

is precisely the class Filt_ω of *filtered* (=*ω-filtered*) ∞-categories (as described in §5 below).

In particular, an ∞-category is filtered if and only if it is both distilled and weakly contractible.

4.6. *Remark.* As we will see, there are many examples of distilled ∞-categories which are not filtered. In particular, all ∞-groupoids are distilled. We give a complete characterization of distilled ∞-categories in §16.

4.7. *Remark.* Any ∞-category which has pushouts is distilled (2.7).

4.8. *Example.* Let F be the free monoid on one generator, regarded as an ∞-category with one object. Then F is distilled, since it has pushouts.

In "geometric" language this means the following: the mapping cylinder construction, which sends a space X with endomorphism $f\colon X \to X$ to $M_f := X \times [0,1]/(x,1) \sim (f(x),0)$, preserves homotopy pullbacks. This can be shown "geometrically" as follows: note that M_f is isomorphic to $T_f/\mathbb{Z}$, where T_f is the "biinfinite mapping telescope" (i.e., the homotopy colimit of $\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots$), which carries an evident free action by $\mathbb{Z}$. The claim follows since $(X, f) \mapsto T_f$ describes a filtered homotopy colimit, and $T_f \mapsto T_f/\mathbb{Z}$ describes a homotopy quotient by a group action, so both steps preserve homotopy pullbacks.

4.9. *Example.* Let $\mathtt{pb}\times := \{\Lambda^2_2, \partial\Delta^1\}$, the "doctrine of pullbacks and binary products", so $\mathrm{Filt}_{\mathtt{pb}\times} = \mathrm{Filt}_{\mathtt{pb}} \cap \mathrm{Filt}_{\times}$. Since any non-empty object of $\mathrm{Filt}_\times$ is sifted and thus contractible (3.12), any non-empty object of $\mathrm{Filt}_{\mathtt{pb}\times}$ is also contractible. Since every object of $\mathrm{Filt}_{\mathtt{pb}\times}$ is also distilled, we learn from (4.5) that

$$\mathrm{Filt}_{\mathtt{pb}\times} = \mathrm{Filt}_\omega \cup \{\varnothing\}.$$

It is easy to see that this is precisely the class of J such that colim_J preserves all limits of *finite and non-empty* diagrams in $\mathcal{S}$.

5. κ-FILTERED ∞-CATEGORIES

Let's briefly recall the following definition [Lur09, 5.3.1.7]. Given a regular cardinal κ, an ∞-category J is **κ-filtered** if every map $f\colon K \to J$ from a κ-small simplicial set K admits an extension along the inclusion $K \subseteq K^{\triangleright}$ to some $\widetilde{f}\colon K^{\triangleright} \to J$.

We can reformulate this as follows: J is κ-filtered if and only if for every $f\colon K \to C$ from a κ-small simplicial set K, the slice $C_{f/}$ is non-empty [Lur09, 5.3.1.11].

Let $\kappa\text{-}\mathtt{sm}$ be the class of ∞-categories which are equivalent to κ-small simplicial sets. Note that κ-$\mathtt{sm}$o is a doctrine, the "doctrine of κ-small limits".

5.1. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$, and κ a regular cardinal. The following are equivalent.*

(1) *$J \in \mathrm{Filt}_{\kappa\text{-}\mathtt{sm}}$.*
(2) *$J \in \mathrm{wFilt}_{\kappa\text{-}\mathtt{sm}}$.*
(3) *J is κ-filtered.*

Proof.
(1) $\Rightarrow$ (2) is by (2.13).
(2) $\Rightarrow$ (3): if $J_{f/}$ is weakly contractible it is certainly non-empty. (Note that the opposite of a κ-simplicial set is also κ-small.)
(3) $\Rightarrow$ (1): this is [Lur09, 5.3.3.3]. $\square$

I'll write Filt_κ for the class of small κ-filtered ∞-categories, so $\mathrm{Filt}_\kappa = \mathrm{Filt}_{\kappa\text{-}\mathtt{sm}} = \mathrm{wFilt}_{\kappa\text{-}\mathtt{sm}}$.

Note that if $\mathcal{U}$ is any doctrine, then $\mathcal{U} \subseteq \kappa\text{-}\mathtt{sm}$ for all sufficiently large regular cardinals κ. Therefore we have the following.

5.2. **Proposition.** *If $\mathcal{U}$ is a doctrine, then $\mathrm{Filt}_{\mathcal{U}} \supseteq \mathrm{Filt}_\kappa$ for all sufficiently large regular cardinals κ.*

We consider the following variant. Let $\kappa\text{-}\mathtt{smconn}$ be the class of ∞-categories which are equivalent to *connected* κ-small simplicial sets. Thus $\kappa\text{-}\mathtt{smconn}$ is the "doctrine of connected κ-small limits".

5.3. **Proposition.** *Let $J \in \mathrm{Cat}_\infty$, and κ a regular cardinal. The following are equivalent.*

(1) *$J \in \mathrm{Filt}_{\kappa\text{-}\mathtt{smconn}}$.*
(2) *$J \in \mathrm{wFilt}_{\kappa\text{-}\mathtt{smconn}}$.*
(3) *J is a coproduct of κ-filtered ∞-categories.*

Proof. (1) $\Rightarrow$ (2) is by (2.13).
(2) $\Rightarrow$ (3): Write $J = \coprod_\alpha J_\alpha$, where the J_α are connected components of the underlying simplicial set of J. Note that each J_α is an ∞-category. Furthermore, for any map $f\colon K \to J_\alpha$, it is straightforward to verify that $(J_\alpha)_{f/} = J_{f/}$. The claim follows from the observation that any map $K \to J$ from a connected K must factor through some connected component J_α.

(3) $\Rightarrow$ (1): If $J = \coprod_{\alpha \in A} J_\alpha$ where each J_αn is κ-filtered, then the J-colimit functor can be factored as a composite

$$\mathrm{Fun}(J, \mathcal{S}) \xrightarrow{\mathrm{LKan}_\pi} \mathrm{Fun}(A, \mathcal{S}) \xrightarrow{\coprod_A} \mathcal{S}.$$

By comparing their right adjoints, we see that LKan_π must be equivalent to a product of colimit functors $\mathrm{colim}_{J_\alpha} \colon \mathrm{Fun}(J_\alpha, \mathcal{S}) \to \mathcal{S}$. The claim follows from the observation that coproducts preserve all connected limits. **(Justify this.)** □

5.4. **Non-regular cardinals.** Suppose κ is an infinite cardinal, but not necessarily regular. Then we can define notions of κ-small simplicial set and κ-filtered ∞-category, using the exact same formulations we gave for regular cardinals. As an immediate consequence, for any infinite (but possibly non-regular) cardinal κ, we have that

$$\mathrm{Filt}_{\kappa\text{-}\mathtt{sm}} \subseteq \mathrm{wFilt}_{\kappa\text{-}\mathtt{sm}} \subseteq \mathrm{Filt}_\kappa.$$

I do not know whether or not these inclusions are strict when κ is not regular.

5.5. *Remark.* For regular cardinals, $\kappa \neq \kappa'$ implies $\mathrm{Filt}_\kappa \neq \mathrm{Filt}_{\kappa'}$. For instance, if $\kappa < \kappa'$, then the smallest ordinal of size κ is κ-filtered but not κ'-filtered.

It turns out that non-regular cardinals are redundant when we are considering κ-$\mathtt{sm}$-filtered ∞-categories. Recall that for any infinite cardinal κ, the successor cardinal κ^+ is regular.

5.6. **Proposition.** *If κ is an infinite cardinal which is not regular, then* $\mathrm{Filt}_{\kappa\text{-}\mathtt{sm}} = \mathrm{Filt}_{\kappa^+\text{-}\mathtt{sm}}$.

Proof. Clearly $\mathrm{Filt}_{\kappa^+\text{-}\mathtt{sm}} \subseteq \mathrm{Filt}_{\kappa\text{-}\mathtt{sm}}$. We need to show that any κ-$\mathtt{sm}$-filtered J is also κ^+-$\mathtt{sm}$-filtered. That is, we need to show that $\mathrm{colim}_J \colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves κ^+-small limits. We prove this using the following lemma (5.7), applied to $\lambda = \kappa^+$. Clearly colim_J preserves pullbacks, since $\omega\text{-}\mathtt{sm} \subseteq \kappa\text{-}\mathtt{sm}$. To show that colim_J preserves κ^+-small products, consider a product indexed by some set S of size less than κ^+. There are two cases.

(1) S has size less than κ, in which case colim_J preserves S-products by hypothesis.
(2) S has size equal to κ (since κ^+ is the successor cardinal). Since κ is not regular while κ^+ is regular, there exists a collection $\{S_i\}_{i \in I}$ of sets with union $S' = \bigcup_{i \in I} S_i$, where $|S_i|, |I| < \kappa$, but $\kappa \leq |S'| < \kappa^+$, whence $|S'| = \kappa$ (since κ^+ is a successor cardinal). Since $|S| = |S'|$ we may assume that $S = S'$. Then a product indexed by S may be computed as composite of products indexed by the S_i and by I, and as colim_J preserves all such products we conclude it preserves products indexed by S, as desired.

□

5.7. **Lemma.** *Let $f \colon A \to B$ be a functor of ∞-categories and assume that λ is a regular cardinal such that that (i) A has λ-small limits and (ii) f preserves pullbacks and λ-small products. Then f preserves λ-small limits.*

Proof. This is (the opposite of) [Lur09, 4.4.2.7]. □

6. Some unsound doctrines

All of the examples of classes $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ we have looked at so far are sound. However, it is easy to give examples of unsound doctrines.

6.1. **Proposition.**

(1) *Let $J \in \mathrm{Cat}_\infty$ be such that $\mathrm{Map}_J(x, y)$ is empty or contractible for every pair of objects x, y in J. Then J is weakly $\{E\}$-filtered, where $E = \Delta^1 \cup_{\partial\Delta^1} \Delta^1$ is the "walking pair of parallel arrows".*
(2) *Let $J \in \mathrm{Cat}_\infty$ be such that $\mathrm{Aut}_J(x)$ is contractible for every object $x \in J$. Then J is weakly $\{S^1\}$-filtered, where $S^1 = B\mathbb{Z}$ is the circle as an ∞-groupoid.*

Proof. Conditions (1) and (2) imply respectively that J has E^{op}-colimits or $(S^1)^{\mathrm{op}}$-colimits, and thus that $J_{f/}$ is weakly contractible for (respectively) any functors $f\colon E^{\mathrm{op}} \to J$ or $f\colon (S^1)^{\mathrm{op}} \to J$. $\square$

6.2. *Example.* The walking cospan $J = \Lambda^2_0$ is weakly $\{E\}$-filtered and weakly $\{S^1\}$-filtered.

6.3. **Proposition.** *The doctrines $\{E\}$ and $\{S^1\}$ are unsound.*

Proof. The walking cospan $J = \Lambda^2_0$ is neither $\{E\}$-filtered or $\{S^1\}$-filtered. For instance, pushouts in $\mathcal{S}$ don't commute with $\mathrm{Map}_{\mathcal{S}}(S^1, -)$. $\square$

6.4. *Remark.* We have that $\mathrm{Filt}_{\{E\}} \supseteq \mathrm{Filt}_{\{S^1\}} \supseteq \mathrm{Filt}_{\omega\text{-}\mathtt{smconn}}$.

...

I do not know any example of $J \in \mathrm{Filt}_{\{E\}}$ which is not a coproduct of filtered ∞-categories.

6.5. *Example.* Let $\mathcal{U} = \{\Lambda^2_2, S^1\}$. The free monoid on one generator F is weakly $\mathcal{U}$-filtered, by (4.8) and (6.1). However, although F is distilled, it is not $\{S^1\}$-filtered, and thus is not $\mathcal{U}$-filtered. For instance, if $f\colon S^1 \times F \to \mathcal{S}$ is a constant functor with value X, then $\mathrm{colim}_F \lim_{S^1} f \to \lim_{S^1} \mathrm{colim}_F f$ is the evident map $\mathrm{Map}(S^1, X) \times S^1 \to \mathrm{Map}(S^1, X \times S^1)$, which is never an equivalence if X is non-empty.

Later we will show that $\mathrm{Filt}_{\mathcal{U}}$ consists exactly of coproducts of ω-filtered ∞-categories (16.14).

7. $\mathcal{U}$-filtered ∞-groupoids

In this section we will show that our notions of $\mathcal{U}$-filtration coincide for ∞-groupoids, and in fact restrict to a *nullity class*. These results (7.3) and (7.11) are summarized by the sequence of equalities

$$\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} = \mathrm{wFilt}_{\mathcal{U}} \cap \mathcal{S} = B(\mathrm{Filt}_{\mathcal{U}}) = \mathrm{Null}_{\mathrm{B}\mathcal{U}}.$$

7.1. **Groupoid completion.** Any ∞-category C has a **groupoid completion**, i.e., a weak equivalence $C \to C'$ to an ∞-groupoid. We will write $BC = C'$ for any choice of groupoid completion of C. We note the following.

7.2. **Lemma.** *For any $C \in \mathrm{Cat}_\infty$, there exists a right anodyne map $j\colon C \to BC$ to a Kan complex.*

Proof. By the small object argument, there exists a factorization $C \xrightarrow{j} C' \xrightarrow{\pi} *$ into a right anodyne map j followed by a right fibration π. Since C' is a right fibrant simplicial set it is actually a Kan complex, and since j is right anodyne it is in particular anodyne and thus a weak equivalence. Thus $j\colon C \to C'$ is a model for the groupoid completion $C \to BC$. $\square$

7.3. **Proposition.** *Let $\mathcal{U} \subseteq \mathrm{Cat}_\infty$. If $J \in \mathrm{Filt}_{\mathcal{U}}$, then $BJ \in \mathrm{Filt}_{\mathcal{U}}$.*

Proof. Let $j\colon J \to BJ$ be a right anodyne map as in (7.2). Since right anodyne maps are cofinal [Lur09, 4.1.1.3], the conclusion follows from (2.17). $\square$

7.4. *Remark.* It is not necessarily the case that if $J \in \mathrm{wFilt}_{\mathcal{U}}$ then $BJ \in \mathrm{wFilt}_{\mathcal{U}}$. For instance, let $J = \Delta^1 \cup_{\partial\Delta^1} \Delta^1$ be the walking pair of parallel arrows, and let $\mathcal{U} = \{B\mathbb{Z}\}$. Then $J \in \mathrm{wFilt}_{\{B\mathbb{Z}\}}$ since J trivially has $B\mathbb{Z}$-limits. However $BJ \notin \mathrm{wFilt}_{\{B\mathbb{Z}\}}$. We can see this by noting that $BJ \approx B\mathbb{Z} \approx B\mathbb{Z}^{\mathrm{op}}$, and that the slice $(B\mathbb{Z})_{\mathrm{id}/}$ of the identity functor of $B\mathbb{Z}$ is empty, and so certainly not weakly contractible.

7.5. **Corollary.** *For any class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$, we have*

$$\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} = B(\mathrm{Filt}_{\mathcal{U}}) := \{\, BJ \mid J \in \mathrm{Filt}_{\mathcal{U}} \,\}.$$

Proof. We have $B(\mathrm{Filt}_{\mathcal{U}}) \subseteq \mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S}$ by (7.3), while $\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} \subseteq B(\mathrm{Filt}_{\mathcal{U}})$ since $B(BJ) \approx BJ$. $\square$

7.6. **Nullity classes.** Let $\mathcal{U} \subseteq \mathcal{S}$ be a class of ∞-groupoids. We say that $X \in \mathcal{S}$ is **$\mathcal{U}$-null** if the tautological map $X \to \mathrm{Map}_{\mathcal{S}}(U, X)$ is an equivalence for all $U \in \mathcal{U}$. We write $\mathrm{Null}_{\mathcal{U}} \subseteq \mathcal{S}$ for the class of $\mathcal{U}$-null ∞-groupoids. When $\mathcal{U}$ is a doctrine, we refer to a such a class as a **nullity class**.

7.7. *Remark.* Farjoun introduces the notion of a nullity class of unbased spaces in [Far96], although there he only considers nullity classes of the form $\mathrm{Null}_{\{\mathrm{U}\}}$, i.e., of singleton sets. In fact, for any set $\mathcal{U}$ of ∞-groupoids there exists a $U \in \mathcal{S}$ such that $\mathrm{Null}_{\mathcal{U}} = \mathrm{Null}_{\{\mathrm{U}\}}$, so the nullity classes we consider here are no more general than Farjoun's. If $\mathcal{U} = \{U_i\}$ contains the empty space as an element, then take $U = \varnothing$. If all elements of $\mathcal{U}$ are non-empty, then take $U := \bigvee U_i$ to be a one-point union of the U_is with respect to some arbitrary choice of basepoints of these spaces.

7.8. **Lemma.** *Let $U \in \mathrm{Cat}_\infty$ and $J \in \mathcal{S}$. Then for any functor $F \colon J \times U \to \mathcal{S}$ there exists a pullback square in $\mathcal{S}$ of the form*

$$\begin{array}{ccc} \operatorname{colim}_J \lim_U F & \longrightarrow & \lim_U \operatorname{colim}_J F \\ \downarrow & & \downarrow \\ J & \longrightarrow & \mathrm{Map}_{\mathcal{S}}(BU, J) \end{array}$$

where the horizontal maps are the tautological ones.

Proof. **[citation needed]** $\square$

Thus, any BU-null ∞-groupoid is also U-filtered.

7.9. *Remark.* Here is a sketch of a "geometric" proof of (7.8). Take U to be a topologically enriched category, and J a topological space. We consider the following data:

(1) The category $\mathrm{Fun}(U, \mathrm{Top})$ of topologically enriched functors, equipped with the projective model structure.
(2) The constant functor $U \to \mathrm{Top}$ with value J, which I will also call J.
(3) An enriched functor $F \colon U \to \mathrm{Top}$ equipped with a fibration $\pi \colon F \to J$ in $\mathrm{Fun}(U, \mathrm{Top})$.
(4) An enriched functor $E \colon U \to \mathrm{Top}$ which is a cofibrant approximation to the terminal object $*$ of $\mathrm{Fun}(U, \mathrm{Top})$.

Then we can form a pullback square of spaces of the form

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Map}(E, F) \\ \downarrow & & \downarrow \\ \mathrm{Map}(*, J) & \longrightarrow & \mathrm{Map}(E, J) \end{array}$$

where "Map" is the space of maps between objects of $\mathrm{Fun}(U, \mathrm{Top})$. The vertical maps of this square are fibrations, so it is a homomtopy pullback. This square is precisely the desired pullback square of the lemma, using the interpretation of $\mathrm{Fun}(U, \mathrm{Top})_{/J}$ as a model for "functors $J \times U \to \mathcal{S}$", that the forgetful functors $\mathrm{Fun}(U, \mathrm{Top})_{/J} \to \mathrm{Fun}(U, \mathrm{Top})$ and $\mathrm{Top}_{/J} \to \mathrm{Top}$ "compute colim_J", and that $\operatorname{colim}_U E \approx BU$. Note that the left-hand arrow is a fibration $P \to J$ whose fiber P_x over $x \in J$ is precisely the homotopy limit of $F_x \colon U \to \mathrm{Top}$, where $F_x(u)$ is the fiber of $F(u)$ over x.

7.10. *Remark.* Here's a sketch of an ∞-categorical proof of (7.8). Let $A \colon U \to \mathcal{S}$ be the constant functor with value J. We use equivalences $\mathcal{S}_{/J} \approx \mathrm{Fun}(J, \mathcal{S})$ and $\mathrm{Fun}(U, \mathcal{S})_{/A} \approx \mathrm{Fun}(U \times J, \mathcal{S})$. The key observation is that the limit functor $\lim_U \colon \mathrm{Fun}(U \times J, \mathcal{S}) \to \mathrm{Fun}(J, \mathcal{S})$ is equivalent to the composite

$$\mathrm{Fun}(U, \mathcal{S})_{/A} \xrightarrow{\lim_U} \mathcal{S}_{/\lim_U A} \xrightarrow{c^*} \mathcal{S}_{/J},$$

where we are using "$\lim_U\colon \mathrm{Fun}(U,\mathcal{S}) \to \mathcal{S}$" and pullback along the tautological map $c\colon J \to \lim_U A$. This leads to a diagram

$$\begin{tikzcd}
\mathrm{Fun}(U,\mathcal{S})_{/A} \arrow[r, "\lim_U"] \arrow[d, "(p_A)_!"'] & \mathcal{S}_{/\lim_U A} \arrow[r, "c^*"] \arrow[d, "(p_{\lim_U A})_!"'] & \mathcal{S}_{/J} \arrow[d, "(p_J)_!"'] \\
\mathrm{Fun}(U,\mathcal{S}) \arrow[r, "\lim_U"'] & \mathcal{S} \arrow[r, equal] & \mathcal{S}
\end{tikzcd}$$

in which the left-hand square commutes, and the right-hand square comes with a Cartesian transformation $(p_J)_! c^* \to (p_{\lim_U A})_!$. Patching gives a Cartesian transformation, which is presumably the tautological one. (Here $f_!$ denotes the evident functor $\mathcal{X}_{/X} \to \mathcal{X}_{/Y}$ associated to $f\colon X \to Y$.)

7.11. **Corollary.** *For any class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$, we have $\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} = \mathrm{wFilt}_{\mathcal{U}} \cap \mathcal{S} = \mathrm{Null}_{\mathrm{B}\mathcal{U}}$, where $B\mathcal{U} := \{\, BU \mid U \in \mathcal{U}\,\}$.*

Proof. We have $\mathrm{Null}_{\mathrm{B}\mathcal{U}} \subseteq \mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} \subseteq \mathrm{wFilt}_{\mathcal{U}} \cap \mathcal{S}$ from (7.8) and (2.13).

Now suppose that $X \in \mathrm{wFilt}_{\mathcal{U}} \cap \mathcal{S}$, so that for all $U \in \mathcal{U}$ and $f\colon U^{\mathrm{op}} \to X$ we have that $X_{f/}$ is weakly contractible. But $X_{f/}$ is weakly equivalent to the fiber of $\mathrm{Map}_{\mathcal{S}}(B(U^{\mathrm{op}})^{\triangleright}, X) \to \mathrm{Map}_{\mathcal{S}}(U^{\mathrm{op}}, X)$ over f. Since $(U^{\mathrm{op}})^{\triangleright}$ is weakly contractible and $U \approx U^{\mathrm{op}}$, this amounts to saying that $X \to \mathrm{Map}_{\mathcal{S}}(BU, X)$ is an equivalence for all $U \in \mathcal{U}$, so $X \in \mathrm{Null}_{\mathrm{B}\mathcal{U}}$ as desired. □

8. Regular classes and filtration classes

Consider a class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ of small ∞-categories. Let $\overline{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ be the class of small ∞-categories U such that $\mathrm{colim}_F\colon \mathrm{Fun}(F,\mathcal{S}) \to \mathcal{S}$ preserves U limits for all $F \in \mathrm{Filt}_{\mathcal{U}}$. I will call $\overline{\mathcal{U}}$ the **regular closure** of $\mathcal{U}$. By definition we have $\mathcal{U} \subseteq \overline{\mathcal{U}} = \overline{\overline{\mathcal{U}}}$ and $\mathrm{Filt}_{\mathcal{U}} = \mathrm{Filt}_{\overline{\mathcal{U}}}$. Note also that

$$\mathcal{U} \subseteq \mathcal{V} \qquad \Longrightarrow \qquad \mathrm{Filt}_{\mathcal{U}} \supseteq \mathrm{Filt}_{\mathcal{V}} \qquad \Longrightarrow \qquad \overline{\mathcal{U}} \subseteq \overline{\mathcal{V}},$$

and that $\overline{U} = \overline{V}$ if and only if $\mathrm{Filt}_U = \mathrm{Filt}_V$.

I will say that a class of small ∞-categories is a **regular class** if it is equal to $\overline{\mathcal{U}}$ for some *doctrine* $\mathcal{U}$, and a **filtration class** if it is equal to $\mathrm{Filt}_{\mathcal{U}}$ for some doctine $\mathcal{U}$. When $\mathcal{U}$ is a doctrine we call $\overline{\mathcal{U}}$ and $\mathrm{Filt}_{\mathcal{U}}$ the regular and filtration classes **generated** by $\mathcal{U}$. Evidently there is a bijective and order reversing correspondence between regular and filtration classes.

8.1. *Remark.* As we have seen (5.6), (5.5), the regular and filtration classes $\overline{\kappa\texttt{-sm}}$ and $\mathrm{Filt}_{\kappa\texttt{-sm}}$ associated to infinite cardinals κ are in fact in bijective correspondence with the *regular* infinite cardinals. Thus, we may view regular classes as a kind of generalization of regular cardinales.

8.2. *Remark.* Every regular/filtration class determines a unique nullity class in ∞-groupoids, by $\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} = \mathrm{Null}_{\mathrm{B}\mathcal{U}}$ (7.11). In particular, for any doctrine $\mathcal{U} \subseteq \mathcal{S}$ of ∞-groupoids we have $\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S} = \mathrm{Null}_{\mathcal{U}}$, so every nullity class arises this way. Thus, we may view filtration classes as a kind of generalization of nullity classes.

The theory of nullication classes is quite rich (see [Far96]), and therefore so is the the theory of regular classes. It thus seems unreasonable to give a complete classification of regular classes.

8.3. *Remark.* Here is a diagram illustrating all doctrines which are subsets of $\{\varnothing, \partial\Delta^1, \Lambda^2_2\}$, together with their associated filtration classes. All of these doctrines are sound: see (5.1), (3.11), (3.14), (3.8), (3.5), (3.12), (4.9).

$$
\begin{tikzcd}
 & {*,\times,\mathrm{pb}} & \\
{*,\times} \arrow[ur, tail] & {*,\mathrm{pb}} \arrow[u, tail] & {\times,\mathrm{pb}} \arrow[ul, tail] \\
{*} \arrow[u, tail] \arrow[ur, tail] & {\times} \arrow[ul, tail] \arrow[ur, tail] & {\mathrm{pb}} \arrow[ul, tail] \arrow[u, tail] \\
 & {\varnothing} \arrow[ul, tail] \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{filtered} \arrow[dl, tail] \arrow[d, equal] \arrow[dr, tail] & \\
\text{sifted} \arrow[d, tail] \arrow[dr, tail] & \text{filtered} \arrow[dl, tail] \arrow[dr, tail] & \text{filtered or } \varnothing \arrow[dl, tail] \arrow[d, tail] \\
\text{weakly contractible} \arrow[dr, tail] & \text{sifted or } \varnothing \arrow[d, tail] & \text{distilled} \arrow[dl, tail] \\
 & \text{all} &
\end{tikzcd}
$$

Compare with the discussion in the introduction to [BJLS15], which in the 1-categorical setting describes analogs to these classes. Note that in that setting "weakly contractible" is replaced by "connected", and "distilled" is replaced by "pseudo-filtered". A 1-category is **pseudo-filtered** its connected components are filtered; these are precisely the 1-categories in $\mathrm{Filt}_{\omega\text{-}\mathtt{smconn}}$, the class of ∞-categories which are filtered for finite connected limits (5.3).

The reader may surmise that a better ∞-categorical analog for this would be $\mathrm{Filt}_{\omega\text{-}\mathtt{smctr}}$, the class of ∞-categories which are filtered for finite *weakly contractible* limits. As we will see, these are precisely the same as distilled ∞-categories (16.15).

8.4. **The minimal regular class.** The smallest regular class is $\overline{\varnothing}$, the class of $U \in \mathrm{Cat}_\infty$ such that $\mathrm{colim}_J\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-limits for all $J \in \mathrm{Cat}_\infty$, or equivalently that $\lim_U\colon \mathrm{Fun}(U, \mathcal{S}) \to \mathcal{S}$ preserves all small colimts. Elements of $\overline{\varnothing}$ are contained in every regular class.

8.5. **Proposition.** *Let $U \in \mathrm{Cat}_\infty$. The following are equivalent.*

(1) *$U \in \overline{\varnothing}$.*
(2) *The Karoubi completion (i.e., idempotent completion) U^+ of U has an initial object.*

Proof. (1) ⇒ (2). The limit functor $\lim_U\colon \mathrm{Fun}(U, \mathcal{S})$ is corepresentable by the terminal object $t \in \mathrm{Fun}(U, \mathcal{S})$. That $\lim_U$ preserves colimits means that t is "completely compact" in the sense of [Lur09, 5.1.6.5], and so is a retract of some corepresentable functor [Lur09, 5.1.6.8]. The full subcategory in $\mathrm{Fun}(U, \mathcal{S}) = \mathrm{PSh}(U^{\mathrm{op}})$ of retracts of corpresentables realizes the Karoubi completion of U^{op}, so $(U^{\mathrm{op}})^+ = (U^+)^{\mathrm{op}}$ has a terminal object, or equivalently U^+ has an initial object.

(2) ⇒ (1). Note that if U itself has an iniial object i, then $\lim_U\colon \mathrm{Fun}(U, \mathcal{S}) \to \mathcal{S}$ is equivalent to evaluation at i, and so preserves all colimits. For the general statement note that the evident restriction functor $\mathrm{Fun}(U^+, \mathcal{S}) \to \mathrm{Fun}(U, \mathcal{S})$ is an equivalence and is compatible with taking U and U^+ colimits. □

8.6. *Example.* The "walking idempotent" Idem is contained in every regular class.

9. $\mathcal{U}$-accessible and weakly $\mathcal{U}$-accessible categories

Given $C \in \mathrm{Cat}_\infty$ and a doctrine $\mathcal{U} \subseteq \mathrm{Cat}_\infty$, we define full subcategories of the presheaf category $\mathrm{PSh}(C)$:

- $\mathrm{Ind}_{\mathcal{U}}(C) \subseteq \mathrm{PSh}(C)$, spanned by presheaves X such that $(C/X) \in \mathrm{Filt}_{\mathcal{U}}$. We call this the subcategory of **$\mathcal{U}$-ind presheaves**.
- $\mathrm{wInd}_{\mathcal{U}}(C) \subseteq \mathrm{PSh}(C)$, spanned by presheaves X such that $(C/X) \in \mathrm{wFilt}_{\mathcal{U}}$. We call this the subcategory of **weakly $\mathcal{U}$-ind presheaves**.

It is clear that $\mathrm{Ind}_{\mathcal{U}}(C) \subseteq \mathrm{wInd}_{\mathcal{U}}(C)$, since $\mathcal{U}$-filtered implies weakly $\mathcal{U}$-filtered (2.13). Also, $\mathrm{Ind}_{\mathcal{U}}(C)$ contains all representable presheaves, since $(C/\rho(c)) \approx C_{/c}$ has a terminal object and so is $\mathcal{U}$-filtered for every $\mathcal{U}$.

Note that if $\mathcal{U}$ is a sound doctrine, then $\mathrm{Ind}_{\mathcal{U}}(C) = \mathrm{wInd}_{\mathcal{U}}(C)$ for all $C \in \mathrm{Cat}_\infty$. In fact the converse is true.

9.1. **Proposition.** *A doctrine $\mathcal{U}$ is sound if and only if $\mathrm{Ind}_{\mathcal{U}}(C) = \mathrm{wInd}_{\mathcal{U}}(C)$ for all $C \in \mathrm{Cat}_\infty$.*

Proof. Take $C = 1$ to be the terminal category. Then for a terminal presheaf $* \in \mathrm{PSh}(C)$ we have $(C/*) \approx C$. Thus C is $\mathcal{U}$-filtered if and only if $* \in \mathrm{Ind}_{\mathcal{U}}(C)$, and weakly $\mathcal{U}$-filtered if and only if $* \in \mathrm{wInd}_{\mathcal{U}}(C)$. □

Say that an ∞-category $\mathcal{A}$ is **$\mathcal{U}$-accessible** if it is equivalent to $\mathrm{Ind}_{\mathcal{U}}(C)$ for some $C \in \mathrm{Cat}_\infty$, and **weakly $\mathcal{U}$-accessible** if it is equivalent to $\mathrm{wInd}_{\mathcal{U}}(C)$ for some $C \in \mathrm{Cat}_\infty$.

9.2. *Example.* Let $\mathcal{U} = \varnothing$ be the trivial doctrine. Then $\mathrm{Ind}_{\varnothing}(C) = \mathrm{wInd}_{\varnothing}(C) = \mathrm{PSh}(C)$.

9.3. *Example.* Let $\mathcal{U}$ be the class of finite discrete ∞-groupoids, i.e., the "doctrine of finite products". Then $\mathrm{Ind}_{\mathcal{U}}(C) = \mathrm{wInd}_{\mathcal{U}}(C)$ consists of all presheaves X such that (C/X) is sifted.

9.4. *Example.* Let $\mathcal{U} = \kappa\text{-}\mathtt{sm}$, the "doctrine of κ-small limits", for some regular cardinal κ. Recall that $\mathrm{Ind}_\kappa(C) \subseteq \mathrm{PSh}(C)$ is defined to be the class of presheaves X which classify a right fibration $D \to C$ where D is κ-filtered [Lur09, 5.3.5.1]. Since $D \approx (C/X)$, we have

$$\mathrm{Ind}_{\kappa\text{-}\mathtt{sm}}(C) = \mathrm{wInd}_{\kappa\text{-}\mathtt{sm}}(C) = \mathrm{Ind}_\kappa(C).$$

Thus, "(weakly) κ-$\mathtt{sm}$-accessible ∞-categories" are exactly the same as "κ-accessible ∞-categories" in the sense of Lurie.

10. Free $\mathcal{U}$-filtered colimit completion

We fix a doctrine $\mathcal{U}$ and a small ∞-category C.

10.1. **Proposition.** *The ∞-category $\mathrm{Ind}_{\mathcal{U}}(C)$ is stable under $\mathcal{U}$-filtered colimits in $\mathrm{PSh}(C)$. (That is, $\mathrm{Ind}_{\mathcal{U}}(C)$ has all $\mathcal{U}$-filtered colimits and the inclusion into $\mathrm{PSh}(C)$ preserves them.)*

Proof. Consider a colimit diagram $X\colon J^\triangleright \to \mathrm{PSh}(C)$, such that J is $\mathcal{U}$-filtered, and such that for each object $j \in J$, the comma category $(C/X(j))$ is $\mathcal{U}$-filtered. We want to show that $(C/X(v))$ is $\mathcal{U}$-filtered, where $v \in J^\triangleright$ is the cone point.

We first observe that, without loss of generality, we may assume that $X(v)$ is a terminal object of $\mathrm{PSh}(C)$. To see this, write $C' := (C/X(v))$ and note that $\mathrm{PSh}(C)_{/X(v)} \approx \mathrm{PSh}(C')$, so that X corresponds to the terminal object of $\mathrm{PSh}(C')$, and the evident functor $X'\colon J^\triangleright \to \mathrm{PSh}(C)_{/X(v)} \approx \mathrm{PSh}(C')$ is a colimit diagram. Furthermore, for each object $j \in J$, we have equivalences $(C'/(X'(j)) \approx (C/X(j))$. Thus, if the special case holds, we may apply it to X', and conclude that $(C'/*) \approx (C/X(v))$ is $\mathcal{U}$-filtered, since both J and all $(C'/X'(j)) \approx (C/X(j))$ are $\mathcal{U}$-filtered.

Now assume that $X(v)$ is a terminal presheaf. We want to show that $(C/X(v)) \approx C$ is $\mathcal{U}$-filtered, i.e., that for any $U \in \mathcal{U}$, the full subcategory $\mathrm{Fun}^{\lim}(U^\triangleleft, \mathcal{S}) \subseteq \mathrm{Fun}(U^\triangleleft, \mathcal{S})$ of limit diagrams is stable under C colimits. Let $F\colon J \to \mathrm{Fun}(U^\triangleleft, \mathcal{S})$ be any functor taking values in the full subcategory $\mathrm{Fun}^{\lim}(U^\triangleleft, \mathcal{S})$, and let $\overline{F}\colon \mathrm{PSh}(C) \to \mathrm{Fun}(U^\triangleleft, \mathcal{S})$ be the left Kan extension on F along the Yoneda functor $\rho\colon C \to \mathrm{PSh}(C)$. We want to show that $\mathrm{colim}_C F \approx \overline{F}(*)$ is in $\mathrm{Fun}^{\lim}(U^\triangleleft, \mathcal{S})$, and hence that C is $\mathcal{U}$-filtered.

For each object $j \in J$ we have the tautological colimit $X(j) \approx \mathrm{colim}_{(C/X(j))} \rho \circ \pi_j^*$, where $\rho\colon C \to \mathrm{PSh}(C)$ is the Yoneda functor and $\pi_j\colon (C/X(j)) \to C$ is the projection. Since the values of $\rho \circ \pi_j^*$ are representable presheaves, the composite functor $\overline{F} \circ \rho \circ \pi_j^*$ takes values in the full subcategory $\mathrm{Fun}^{\lim}(U^\triangleleft, \mathcal{S})$. Since $(C/X(j))$ is $\mathcal{U}$-filtered, the colimit of $\overline{F} \circ \rho_i$ also takes values in $\mathrm{Fun}^{\lim}(U^\triangleleft, \mathcal{S})$. But

$$\mathrm{colim}_{(C/X(j))} \overline{F} \circ \rho \circ \pi_j^* \approx \overline{F}(\mathrm{colim}_{(C/X(j))} \rho \circ \pi_j^*) \approx \overline{F}(X(j)),$$

so $\overline{F}(X(j)) \in \mathrm{Fun}^{\lim}(U^{\triangleleft}, \mathcal{S})$.

Thus the restriction $F|_J$ of $\overline{F} \circ X \colon J^{\triangleright} \to \mathrm{Fun}(U^{\triangleleft}, \mathcal{S})$ to $J \subseteq J^{\triangleright}$ takes values in $\mathrm{Fun}^{\lim}(U^{\triangleleft}, \mathcal{S})$, and since J is $\mathcal{U}$-filtered so does the colimit of $F|_J$ its colimit. But $\overline{F}$ is colimit preserving, so

$$\mathrm{colim}_J\, \overline{F} \circ X \approx \overline{F}(X(v)) \approx \overline{F}(*) \approx \mathrm{colim}_J\, F,$$

so the colimit of F is a limit diagram as claimed. □

10.2. **Corollary.** *The ∞-category $\mathrm{Ind}_{\mathcal{U}}(C)$ is the smallest subcategory of $\mathrm{PSh}(C)$ containing the essential image of $\rho\colon C \to \mathrm{PSh}(C)$ and which is stable under $\mathcal{U}$-filtered colimits.*

Proof. Straightforward from the previous proposition, and the fact that every object of $\mathrm{Ind}_{\mathcal{U}}(C)$ is a $\mathcal{U}$-filtered colimit of representable presheaves. □

As a consequence, the restriction of Yoneda to a functor $\rho\colon C \to \mathrm{Ind}_{\mathcal{U}}(C)$ exhibits the **free $\mathcal{U}$-filtered colimit completion** of C.

10.3. **Corollary.** *If $\mathcal{A}$ is any ∞-category with all $\mathcal{U}$-filtered colimits, restriction along ρ defines an equivalence*

$$\mathrm{Fun}(\mathrm{Ind}_{\mathcal{U}}(C), \mathcal{A}) \supseteq \mathrm{Fun}^{\mathrm{Filt}_{\mathcal{U}}\,\mathrm{colim}}(\mathrm{Ind}_{\mathcal{U}}(C), \mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}(C, \mathcal{A}).$$

from the full subcategory of functors which preserve $\mathcal{U}$-filtered colimits.

Proof. (Compare [Lur09, 5.3.5.10].) This is a consequence of [Lur09, 5.3.6.2]. □

11. $\mathcal{U}$-compact objects

Let $\mathcal{A}$ be an ∞-category which is closed under $\mathcal{U}$-filtered colimits. We say that an object $A \in \mathcal{A}$ is **$\mathcal{U}$-compact** if the functor

$$\mathrm{Map}_{\mathcal{A}}(A, -)\colon \mathcal{A} \to \mathcal{S}$$

corepresented by A preserves $\mathcal{U}$-filtered colimits.

For an ∞-category $\mathcal{A}$ which is closed under $\mathcal{U}$-filtered colimits, we write $\mathcal{A}^{\mathcal{U}-\mathrm{cpt}} \subseteq \mathcal{A}$ for the full subcategory of $\mathcal{U}$-compact objects.

11.1. **Proposition.** *$\mathrm{Ind}_{\mathcal{U}}(C)^{\mathcal{U}-\mathrm{cpt}}$ is precisely the idempotent completion of C in $\mathrm{PSh}(C)$.*

Proof. (This is basically the proof of [Lur09, 5.4.2.4].) We know that $\mathrm{Filt}_{\mathcal{U}}$ contains the walking idempotent (3.3), so $\mathcal{A} := \mathrm{Ind}_{\mathcal{U}}(C)$ is idempotent complete, and thus contains a full subcategory which is the idempotent completion C^+ of C. Furthermore, it is clear that $C^+ \subseteq C' := \mathcal{A}^{\mathcal{U}-\mathrm{cpt}}$, since a retract of a $\mathcal{U}$-compact object must also be $\mathcal{U}$-compact.

It remains to show that if $X \in C'$ then X is a retract of a representable. Every object $X \in \mathcal{A}$ is a colimit of the composite $(C/X) \xrightarrow{\pi} C \xrightarrow{\rho} A$, or in other words, the composite τ of

$$(C/X)^{\triangleright} \xrightarrow{i^{\triangleright}} (A_{/X})^{\triangleright} \xrightarrow{r} A$$

is a colimit, where $i\colon (C/X) \to A_{/X}$ is the tautological map and r is the "natuaral functor" **(what is this)**? Furthermore, (C/X) is $\mathcal{U}$-filtered.

The corepresentable functor $\mathrm{Map}_{\mathcal{A}}(X, -)\colon \mathcal{A} \to \mathcal{S}$ classifies the left fibration $q\colon \mathcal{A}_{X/} \to \mathcal{A}$ (the forgetful map). Since X is $\mathcal{U}$-compact the functor $\mathrm{Map}_{\mathcal{A}}(X, -)$ preserves the tautological colimit τ. Therefore by [Lur09, 3.3.4.5], the induced inclusion

$$(C/X) \times_{\mathcal{A}} \mathcal{A}_{X/} \rightarrowtail (C/X)^{\triangleright} \times_{\mathcal{A}} \mathcal{A}_{X/}$$

is a weak equivalence of simplicial sets. The target has a tautological vertex (v, id_X) (corresponding to the identity of X as an object of $\mathcal{A}_{X/}$ and the cone point of $(C/X)^{\triangleright}$). Therefore there exists a vertex in the source in the same path component. Such a vertex corresponds to a commutative diagram of the form

$$\begin{tikzcd} & \rho(c) \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "\mathrm{id}_X"'] & & X \end{tikzcd}$$

in $\mathcal{A}$ for some $c \in C$, so the claim is proved. ☐

The following says that $\mathcal{A}^{\mathcal{U}-\mathrm{cpt}}$ is stable under $\overline{\mathcal{U}}^{\mathrm{op}}$-colimits which exist in $\mathcal{A}$.

11.2. **Proposition.** *Let $\mathcal{A}$ be an ∞-category closed under $\mathcal{U}$-filtered colimits. If $f\colon U^{\mathrm{op}} \to \mathcal{A}^{\mathcal{U}-\mathrm{cpt}}$ is a functor with $U \in \overline{\mathcal{U}}$, and if f has a colimit X in $\mathcal{A}$, then $X \in A^{\mathcal{U}-\mathrm{cpt}}$.*

Proof. (Compare [5.3.4.15].) Let $g\colon C \to A$ be a functor with $C \in \mathrm{Filt}_{\mathcal{U}}$. We have

$$\begin{aligned} \mathrm{Map}_A(X, \mathrm{colim}_C\, g) &\approx \mathrm{Map}_A(\mathrm{colim}_{U^{\mathrm{op}}} f, \mathrm{colim}_C\, g) \\ &\approx \lim\nolimits_{u\in U} \mathrm{Map}_A(f(u), \mathrm{colim}_C\, g) \\ &\approx \lim\nolimits_{u\in U} \mathrm{colim}_{c\in C} \mathrm{Map}_A(f(u), g(c)) && \text{since } f(u) \in A^{\mathcal{U}-\mathrm{cpt}}, \\ &\approx \mathrm{colim}_{c\in C} \lim\nolimits_{u\in U} \mathrm{Map}_A(f(u), g(c)) && \text{since } U \in \overline{\mathcal{U}} \text{ and } C \in \mathrm{Filt}_{\mathcal{U}}, \\ &\approx \mathrm{colim}_{c\in C} \mathrm{Map}_A(\mathrm{colim}_{u\in U} f(u), g(c)) \\ &\approx \mathrm{colim}_{c\in C} \mathrm{Map}_A(X, g(c)), \end{aligned}$$

so $X \in A^{\mathcal{U}-\mathrm{cpt}}$. ☐

11.3. **Proposition.** *Let $\mathcal{A}$ be an ∞-category which has all $\mathcal{U}$-filtered colimits. Let C be a small ∞-category and $\mathcal{U} \subseteq \mathrm{Cat}_\infty$. Let $F\colon \mathrm{Ind}_{\mathcal{U}}(C) \to \mathcal{A}$ be a functor which preserves $\mathcal{U}$-filtered colimits, and let $f := F \circ \rho\colon C \to \mathcal{A}$ be its restriction along Yoneda.*

(1) *If f is fully faithful and $f(C) \subseteq \mathcal{A}^{\mathcal{U}-\mathrm{cpt}}$, then F is fully faithful.*
(2) *F is an equivalence iff*
   (i) *f is fully faithful.*
   (ii) *$f(C) \subseteq \mathcal{A}^{\mathcal{U}-\mathrm{cpt}}$.*
   (iii) *The objects $\{f(c)\}_{c\in C}$ generate $\mathcal{A}$ under $\mathcal{U}$-filtered colimits.*

Proof. Same as the proof of [Lur09, 5.3.5.11]. ☐

12. Flat presheaves

Any $X \in \mathrm{PSh}(C)$ gives rise to a functor $\widehat{X}\colon \mathrm{Fun}(C, \mathcal{S}) \to \mathcal{S}$, by $\widehat{X} = \mathrm{LKan}_\rho X$, using $\rho\colon C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathcal{S})$. Note that $\widehat{X}(\rho^{\mathrm{op}}(c)) \approx X(c)$ for $c \in C$.

Say that X is **$\mathcal{U}$-flat** if $\widehat{X}$ preserves all $\mathcal{U}$-limits. We write $\mathrm{Flat}_{\mathcal{U}}(C) \subseteq \mathrm{PSh}(C)$ for the full subcategory spanned by all $\mathcal{U}$-flat presheaves.

12.1. **Proposition.** *If (C/X) is $\mathcal{U}$-filtered, then $\widehat{X}$ is $\mathcal{U}$-flat.*

Proof. We can compute $\widehat{X}$ as the composite of

$$\mathrm{Fun}(C, \mathcal{S}) \xrightarrow{\pi^*} \mathrm{Fun}((C/X), \mathcal{S}) \xrightarrow{\mathrm{colim}_{(C/X)}} \mathcal{S},$$

where $\pi\colon (C/X) \to C$ is the forgetful functor. Thus if (C/X) is $\mathcal{U}$-filtered then the composite preserves $\mathcal{U}$-limits. ☐

The following is an ∞-categorical version of [ABLR02, 2.4, (ii) $\Rightarrow$ (iv)].

12.2. **Proposition.** *Let $p\colon D \to C$ be a right fibration classified by a functor $X\colon C^{\mathrm{op}} \to \mathcal{S}$. If X is $\mathcal{U}$-flat, then $D_{f/}$ is weakly contractible for every functor $f\colon U^{\mathrm{op}} \to D$ with $U \in \mathcal{U}$.*

12.3. **Corollary.** *If X is $\mathcal{U}$-flat, then (C/X) is weakly $\mathcal{U}$-filtered.*

Proof. Take $D := (C/X)$ and p the evident projection. $\square$

Taking (12.1) and (12.3) together with (9.1) gives the following.

12.4. **Proposition.** *For any $C \in \mathrm{Cat}_\infty$ and class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$, we have*

$$\mathrm{Ind}_\mathcal{U}(C) \subseteq \mathrm{Flat}_\mathcal{U}(C) \subseteq \mathrm{wInd}_\mathcal{U}(C).$$

Furthermore, $\mathcal{U}$ is sound if and only if $\mathrm{Ind}_\mathcal{U}(C) = \mathrm{Flat}_\mathcal{U}(C) = \mathrm{wInd}_\mathcal{U}(C)$.

12.5. *Remark.* In [ABLR02], a category is said to be "$\mathcal{U}$-accessible" if it is equivalent to some $\mathrm{Flat}_\mathcal{U}(C)$. (More precisely, they state the 1-categorical analogue, involving functors to sets.) I am not following this choice of language here.

Before giving the proof of (12.2) I'll describe the idea informally. Recall that $D_{f/}$ is weakly equivalent to $\mathrm{colim}_{d'\in D} \lim_{u\in U} \mathrm{Map}_D(f^{\mathrm{op}}(u), d')$, the colimit of the limit of corepresentable functors indexed by f.

The first idea is to note that for each pair of objects d, d' in D we have a pullback square

$$\begin{tikzcd} \mathrm{Map}_D(d,d') \arrow[r] \arrow[d] & \mathrm{Map}_C(p(d),p(d')) \arrow[d] \\ * \arrow[r, "\lambda_d"'] & X(p(d)) \end{tikzcd}$$

of ∞-groupoids, which is the "formula" for spaces of maps in D in terms of X. These squares are natural in $(d,d') \in D^{\mathrm{op}} \times D$, though note that the bottom row only depends on the first variable d.

Next we can (i) restrict along $f^{\mathrm{op}}\colon U \to D^{\mathrm{op}}$ in the first variable and take limits in U, which preserves all the pullbacks, and then (ii) take the colimit of the material in the top row with respect to $d' \in D$, giving rise to a commutative square

$$\begin{tikzcd} \mathrm{colim}_{d'\in D} \lim_{u\in U} \mathrm{Map}_D(f^{\mathrm{op}}(u),d') \arrow[r] \arrow[d] & \mathrm{colim}_{d'\in D} \lim_{u\in U} \mathrm{Map}_C(p(f^{\mathrm{op}}(u)),p(d')) \arrow[d, "\gamma"] \\ * \arrow[r] & \lim_{u\in U} X(p(f^{\mathrm{op}}(u))) \end{tikzcd}$$

Because $\mathcal{S}$ is an ∞-topos, the descent property implies that this is also a pullback square. Thus if we can show γ is an equivalence, then $\mathrm{colim}_{d'\in D} \lim_{u\in U} \mathrm{Map}_D(f^{\mathrm{op}}(u), d')$ is contractible, as desired.

The map γ is an equivalence because X is $\mathcal{U}$-flat. To see this, note that if $G\colon C \to \mathcal{S}$ is given by $G(c) := \lim_{u\in U} \mathrm{Map}_C(p(f^{\mathrm{op}}(u)), c)$, then $\mathrm{colim}_D G \circ p \approx \widehat{X}(G)$. If X is $\mathcal{U}$-flat then we compute

$$\widehat{X}(G) \approx \widehat{X}(\lim_{u\in U} \rho(p(f^{\mathrm{op}}(u))) \approx \lim_{u\in U} \widehat{X}(\rho(p(f^{\mathrm{op}}(u))) \approx \lim_{u\in U} X(p(f^{\mathrm{op}}(u)),$$

where $\widehat{X}\colon \mathrm{Fun}(C,\mathcal{S}) \to \mathcal{S}$ is the left Kan extension of X, $\rho\colon C^{\mathrm{op}} \to \mathrm{Fun}(C,\mathcal{S})$ is the Yoneda functor, and we use the equivalence $X \approx \widehat{X} \circ \rho$.

12.6. **Ideas for the proof.**

12.7. **Lemma.** *Let $p\colon D \to C$ be a right fibration classified by a functor $F\colon C^{\mathrm{op}} \to \mathcal{S}$. Consider the composite of the functors*

$$C^{\mathrm{op}} \times D \xrightarrow{\mathrm{id}\times p} C^{\mathrm{op}} \times C \xrightarrow{\mathrm{Map}_C} \mathcal{S}$$

and form the adjoint $\phi\colon D \to \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$. Then there is an equivalence $\mathrm{colim}_D\, \phi \approx F$.

Given a functor $p\colon D \to C$, we should get a natural transformation

$$\lambda\colon \operatorname{Map}_D \to \operatorname{Map}_C \circ (p^{\mathrm{op}} \times p)$$

of functors $D^{\mathrm{op}} \times D \to \mathcal{S}$. This adjoints to a functor

$$\widetilde{\lambda}\colon D \to \operatorname{Fun}(\Delta^1, \operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S})).$$

Now suppose p is a right fibration. Then the following should be true:

- $\widetilde{\lambda}$ lands in the (non-full) subcategory $\operatorname{Fun}^{\square}(\Delta^{,} \operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S}))$, of morphisms which are pullback squares in $\operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S})$.
- The colimit of $\widetilde{\lambda}$ in $\operatorname{Fun}(\Delta^1, \operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S}))$ with respect to D should equivalent to a map

  $$\overline{\lambda}\colon * \to F \circ p$$

  in $\operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S})$, where $F\colon C^{\mathrm{op}} \to \mathcal{S}$ is the functor which classifies p, and $\overline{\lambda}$ corresponds to the "tautological section" of $F \circ p$ (which classifies a projection $D \times_C D \to D$).

Since $\operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S})$ is an ∞-topos, we know that $\operatorname{Fun}^{\square}(\Delta^1, \operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S}))$ has colimits and $\gamma\colon \operatorname{Fun}^{\square}(\Delta^1, \operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S})) \to \operatorname{Fun}(\Delta^1, \operatorname{Fun}(D^{\mathrm{op}}, \mathcal{S}))$ preserves colimits.

13. Limit preserving presheaves

Consider a class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ and a small ∞-category $C \in \mathrm{Cat}_\infty$. Suppose that C^{op} has all $\mathcal{U}$-limits (or equivalently, that C has all $\mathcal{U}^{\mathrm{op}}$-colimits). In this case, let $\operatorname{Lim}_{\mathcal{U}}(C) \subseteq \operatorname{PSh}(C)$ denote the full subcategory of presheaves spanned by $X\colon C^{\mathrm{op}} \to \mathcal{S}$ which preserve all $\mathcal{U}$-limits. This is the full subcategory of $\mathcal{U}$-limit preserving presheaves.

13.1. *Remark.* If $\mathcal{U}$ is a doctrine, then $\operatorname{Lim}_{\mathcal{U}}(C)$ is a presentable ∞-category [?].

13.2. **Proposition.** *Suppose $C \in \mathrm{Cat}_\infty$ has all $\mathcal{U}$-limits. Then any $\mathcal{U}$-flat functor $X\colon C^{\mathrm{op}} \to \mathcal{S}$ preserves all $\mathcal{U}$-limits.*

Proof. Immediate from the observation that there exists a commutative diagram of functors of the form

$$\begin{tikzcd} C^{\mathrm{op}} \arrow[d, "\rho"'] \arrow[dr, "X"] & \\ \operatorname{Fun}(C, \mathcal{S}) \arrow[r, "\widehat{X}"'] & \mathcal{S} \end{tikzcd}$$

and the fact that the Yoneda functor ρ preserves all limits which exist in C^{op}. $\square$

13.3. **Lemma.** *Let $p\colon D \to C$ be a left fibration classified by some $f\colon C \to \mathcal{S}$. If C has all $\mathcal{U}$-limits and f preserves $\mathcal{U}$-limits, then D has all $\mathcal{U}$-limits and p preserves $\mathcal{U}$-limits.*

Proof. Because f classifies p, there exists a homotopy pullback square in the Joyal model structure of the form

$$\begin{tikzcd} D \arrow[r] \arrow[d, "p"'] & \mathcal{S}_* \arrow[d, "q"] \\ C \arrow[r, "f"'] & \mathcal{S} \end{tikzcd}$$

Here q is the evident forgetful functor from pointed ∞-groupoids to ∞-groupoids. We know that both $\mathcal{S}$ and $\mathcal{S}_*$ have small limits, and q preserves all limits. The conclusion is immediate from (the opposite of) [Lur09, 5.4.5.5]. $\square$

13.4. **Corollary.** *If C has all $\mathcal{U}$-limits and $X\colon C^{\mathrm{op}} \to \mathcal{S}$ preserves all $\mathcal{U}$-limits, then (C/X) is weakly $\mathcal{U}$-filtered.*

Proof. By (**??**), $(C/X)^{\mathrm{op}}$ has all $\mathcal{U}$-limits, since $(C/X)^{\mathrm{op}} \to C^{\mathrm{op}}$ is a left fibration classifying X. $\square$

13.5. **Corollary.** *Let $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and suppose C is a small ∞-category such that C^{op} has all $\mathcal{U}$-limits. Then we have inclusions*

$$\mathrm{Ind}_{\mathcal{U}}(C) \subseteq \mathrm{Flat}_{\mathcal{U}}(C) \subseteq \mathrm{Lim}_{\mathcal{U}}(C) \subseteq \mathrm{wInd}_{\mathcal{U}}(C).$$

In particular, if $\mathcal{U}$ is sound, then $\mathrm{Ind}_{\mathcal{U}}(C) = \mathrm{Flat}_{\mathcal{U}}(C) = \mathrm{Lim}_{\mathcal{U}}(C) = \mathrm{wInd}_{\mathcal{U}}(C)$.

13.6. *Remark.* It is a consquence of [Lur09, 5.5.4.18–19] that the inclusion $\mathrm{Lim}_{\mathcal{U}}(C) \subseteq \mathrm{PSh}(C)$ is accessible and admits a left adjoint, and so $\mathrm{Lim}_{\mathcal{U}}(C)$ is a presentable ∞-category.

13.7. *Example* (Non-abelian derived categories). Suppose $C \in \mathrm{Cat}_\infty$ has finite coproducts. Then since the doctrine $\mathtt{fin}\times$ of finite products is sound, we have

$$\mathrm{Ind}_{\mathtt{fin}\times}(C) = \mathrm{Flat}_{\mathtt{fin}\times}(C) = \mathrm{Lim}_{\mathtt{fin}\times}(C) = \mathrm{wInd}_{\mathtt{fin}\times}(C).$$

These are precisely the "non-abelian derived categories" of [Lur09, 5.5.8], which are there denoted $\mathcal{P}_\Sigma(C)$. We have thus reproved a few of the results given there, including that $\mathrm{Lim}_{\mathtt{fin}\times}(C)$ is the free sifted-colimit-completion of C [Lur09, 5.5.8.15] .

14. Flat presheaves are accessible

14.1. **Proposition.** *For any doctrine $\mathcal{U}$ and small ∞-category C, the full subcategory $\mathrm{Flat}_{\mathcal{U}}(C)$ of $\mathrm{PSh}(C)$ is accessibly embedded.*

Proof. It suffices to consider the case of $\mathcal{U} = \{U\}$. Let $A = \mathrm{Fun}(U^{\triangleright}, \mathcal{S})$, and let $B \subseteq A$ denote the full subcategory spanned by limit cones. I claim that B is accessibly embedded in A, so that B is λ-accessible and the inclusion $B \to A$ is λ-accessible.

Given a functor $F\colon C \to B$, let $\widehat{F}\colon \mathrm{PSh}(C) \to A$ be any choice of colimit preserving extension of $C \xrightarrow{F} B \subseteq A$. Thus $X \in \mathrm{Flat}_{\mathcal{U}}(\mathcal{C})$ iff $\widehat{F}(X) \in B$ for all F.

... $\square$

BTW, here's my proof: Let $A = \mathrm{Fun}(D^{\triangleleft}, \mathcal{S})$, and let $B \subseteq A$ be the full subcategory spanned by limit cones. Then you can show that B is accessibly embedded in A. Given an $F\colon C \to B$, we can extend to a colimit preserving $\widetilde{F}\colon \mathrm{Psh}(C) \to A$. A presheaf X is flat iff $\widetilde{F}(X) \in B$ for all F. But $\mathrm{Fun}(C, B)$ is accessible, so we can write every F as a sufficiently filtered colimit of some set $\{F_i\}$ of functors, and thus X is flat iff $\widetilde{F}_i(X) \in B$ for all F_i. So flat presheaves are the pullback of $\mathrm{Psh}(C) \to \prod_i A \leftarrow \prod_i B$, a diagram of accessible functors.

15. Colimit closure properties of regular and filtration classes

Condsider a class $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ (not necessarily a doctrine). We have the following.

15.1. **Proposition.**

(1) *The full subcategory $\mathrm{Filt}_{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ is stable under $\mathcal{U}$-filtered colimits.*
(2) *The full subcategory $\overline{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ is stable under $\overline{\mathcal{U}}^{\mathrm{op}}$-colimits.*

The proof of these uses the following decomposition theorem.

15.2. **Proposition** ([HY17, 2.5]). *Let $K\colon I \to \mathrm{Cat}_\infty$ be a functor from a small ∞-category I, with $\overline{K}$. Let C be an ∞-category which has (i) all I-colimits, and (ii) all K_a-colimits for all objects a of I. Then C has all $\overline{K}$-colimits. Furthermore, such colimits are computed as a composite*

$$\mathrm{Fun}(\overline{K}, C) \approx \lim_I \mathrm{Fun}(K, C) \xrightarrow{G^I} \mathrm{Fun}(I, C) \xrightarrow{\mathrm{colim}_I} C,$$

where for an object $\{f_a\}_{a \in K}$ in $\lim_I \mathrm{Fun}(K, C)$, the value of $G^I(\{f_a\})$ at an object $a \in I$ is equivalent to $\mathrm{colim}_{K_a} f_a$.

In other words, we have $\operatorname{colim}_{\overline{K}} f \approx \operatorname{colim}_{a\in I} \operatorname{colim}_{K_a} f|_{K_a}$ when all such colimits exist in C. Likewise, by replacing C with C^{op}, we have that $\lim_{\overline{K}} f \approx \lim_{a\in I^{\mathrm{op}}} \lim_{K_a^{\mathrm{op}}} f|_{K_a}$ when all such limits exist in C.

Proof of (15.1). (1) Apply (15.2) to $C := \mathrm{Fun}^{\lim}(U^{\lhd}, \mathcal{S}) \subseteq \mathrm{Fun}(U^{\lhd}, \mathcal{S})$, the full subcategory of limit cones, for all $U \in \overline{\mathcal{U}}$. (2) Apply (15.2) to $C^{\mathrm{op}} := \mathrm{Fun}^{\mathrm{colim}}(J^{\rhd}, \mathcal{S}) \subseteq \mathrm{Fun}(J^{\rhd}, \mathcal{S})$, the full subcategory of colimit cones, for all $J \in \mathrm{Filt}_{\mathcal{U}}$. □

15.3. *Remark.* It seems likely to me that in (15.1) we should be able to replace "colimit" with "oplax colimit" in the sense of [GHN17].

15.4. *Remark.* Note that if $\mathcal{U}$ is a doctrine, then $\mathrm{Filt}_{\mathcal{U}} \supseteq \mathrm{Filt}_{\kappa}$ for all sufficiently large regular cardinals κ, and so the full subcategory $\mathrm{Filt}_{\mathcal{U}} \subseteq \mathrm{Cat}_{\infty}$ is stable under κ-filtered colimits for such κ. Can it be shown in general that $\mathrm{Filt}_{\mathcal{U}}$ is an accesssible category?

16. Characterization of distilled ∞-categories

We show that, in a certain sense, the distilled ∞-categories are precisely those which are a "bundle of filtered ∞-categories over an ∞-groupoid".

16.1. **∞-categories as colimits of ∞-groupoid indexed diagrams.** Given any functor $p\colon C \to X$ and object $x \in X$, we write $C/x := C \times_X X_{/x}$, which fits in the pullback square

$$\begin{tikzcd} C/x \arrow[r, "\pi"] \arrow[d, "q"'] & C \arrow[d, "p"] \\ X_{/x} \arrow[r, "\gamma"'] & X \end{tikzcd}$$

We will always assume in the following that X is an ∞-groupoid, whence γ is a Kan fibration.

16.2. **Proposition.** *If $p\colon C \to X$ is a functor to an ∞-groupoid X, then C is equivalent to the colimit of a functor $f\colon X \to \mathrm{Cat}_{\infty}$, whose value at each object $x \in X$ is equivalent to C/x.*

Proof. Factor p as $C \to C' \to X$ where the first map is an equivalence and the second an isofibration. The pullback in simplicial sets along $\gamma\colon X_{/x} \to X$ is a homotopy pullback in the Joyal model structure, so $C/x \to C'/x$ is an equivalence. Thus without loss of generality we may assume that p is an isofibration, by replacing C with C'.

Since X is an ∞-groupoid, an edge in C is p-coCartesian if and only if it is an isomorphism [Lur09, ??]. Since p is an isofibration it is therefore a coCartesian fibration. Now we appeal to [Lur09, 3.3.4.3], which asserts that if the coCartesian fibration p is classified by a functor $f\colon X \to \mathrm{Cat}_{\infty}$, then the colimit of f is Cat_{∞} is equivalent to the ∞-category obtained by formally inverting all p-coCartesian edges in C (or rather, it asserts "Cartesian" version of these claims). Since the p-Cartesian edges of C are just the isomorphisms in C, we just have that $\operatorname{colim}_X f \approx C$, as desired

(It is worthwhile spelling this out more carefully in our context. Lurie [Lur09, 3.3.4.3] asserts that the colimit of f is equivalent to $C^{\natural}$ in the model category $\mathrm{Set}_{\Delta}^{+}$ of marked simplicial sets, where $C^{\natural}$ is the simplical set C with its p-Cartesian edges marked. As we have noted, these are precisely the isomorphisms of C, and thus precisely the q-Cartesian edges of $q\colon C \to \Delta^0$. Therefore, $C^{\natural}$ is a fibrant object of $\mathrm{Set}_{\Delta}^{+}$ [Lur09, 3.1.4.1], and therefore corresponds to C under the Quillen equivalence between $\mathrm{Set}_{\Delta}^{+}$ and the Joyal model structure on Set_{Δ} [Lur09, 3.1.5.3].) □

16.3. **Weak filteredness for classes of weakly contractible ∞-categories.**

16.4. **Lemma.** *Let $p\colon C \to X$ any functor from an ∞-category to an ∞-groupoid, and x and object of X. For every functor $f\colon V \to C/x$ from a weakly contractible simplicial set V, the induced functor on slices*

$$(C/x)_{f/} \to C_{f\pi/}$$

is a trivial fibration of simplicial sets.

Proof. Formation of slices is limit preserving as a functor $\mathrm{sSet}_{V/} \to \mathrm{sSet}$, so

$$\begin{tikzcd}
(C/x)_{f/} \arrow[r, "\pi'"] \arrow[d] & C_{\pi f/} \arrow[d] \\
(X_{/x})_{qf/} \arrow[r, "\gamma'"'] & X_{\gamma q f/}
\end{tikzcd}$$

is a pullback square. The map γ' factors as a composite of two maps

$$(X_{/x})_{qf/} \to (X_{\gamma qf/}) \times_X X_{/x} \to X_{\gamma qf/}.$$

The first is formally a Kan fibration by verifying the lifting property, while the second is a base change of γ which is also seen to be a left fibration and hence a Kan fibration since X is a Kan complex. Thus γ' is a Kan fibration.

Furthermore, both the source and target of γ' are weakly contractible, since they are both slices of a functor from weakly contractible V to a Kan complex. Thus γ' is a trivial fibration, and hence so is π' as desired. $\square$

16.5. **Lemma.** *Let $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ be a class of weakly contractible ∞-categories. Let C be an ∞-category, and let $p\colon C \to X$ a functor to an ∞-groupoid X. The following are equivalent.*

(1) *$C \in \mathrm{wFilt}_{\mathcal{U}}$.*
(2) *For all objects x of X, we have $C/x \in \mathrm{wFilt}_{\mathcal{U}}$.*

Proof. (1) $\Rightarrow$ (2). Immediate from (16.4), applied to $V := U^{\mathrm{op}}$ for every $U \in \mathcal{U}$ and every $f\colon U^{\mathrm{op}} \to C/x$.

(2) $\Rightarrow$ (1). Consider $g\colon U^{\mathrm{op}} \to C$ for some $U \in \mathcal{U}$. Since U^{op} is weakly contractible it is non-empty, so pick any object u of U^{op} and set $x := g(u)$. As $\gamma\colon X_{/x} \to X$ is a Kan fibration and $u\colon \Delta^0 \to U$ is a monomorphism and weak equivalence, we can construct a lift of pg to a map $U^{\mathrm{op}} \to X_{/x}$, and hence obtain $f\colon U^{\mathrm{op}} \to C/x$ such that $\pi f = g$. We conclude that $C_{g/} = C_{f\pi/}$ is weakly contractible by the hypothesis and (16.4). $\square$

Now we apply this in the special case of a groupoid completion functor $p\colon C \to BC$. As before, we write C/x for the pullback of p along $BC_{/x} \to BC$.

16.6. **Proposition.** *Let $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ be a class of weakly contractible ∞-categories. Then $C \in \mathrm{wFilt}_{\mathcal{U}}$ if and only if $C/x \in \mathrm{wFilt}_{\mathcal{U}} \cap \mathrm{Filt}_{\mathtt{term}}$ for all objects x of BC.*

Proof. Immeditate from (16.5), together with the observation that C/x is necessarily weakly contractible (i.e., an object of $\mathrm{Filt}_{\mathtt{term}}$) when p is a weak equivalence. $\square$

16.7. **Proposition.** *If $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ is a class of weakly contractible ∞-categories, and if $C \in \mathrm{wFilt}_{\mathcal{U}}$, then C is equivalent to the colimit of some functor $f\colon X \to \mathrm{Cat}_\infty$, where X is a groupoid completion of J and f takes values in the full subcategory $\mathrm{wFilt}_{\mathcal{U}} \cap \mathrm{Filt}_{\mathtt{term}}$.*

Proof. Let $p\colon C \to X$ be a groupoid completion. According to (16.2), J is a colimit of a functor $f\colon X \to \mathrm{Cat}_\infty$, whose value $f(x)$ at an object $x \in X$ is equivalent to C/x. By (16.5) each $C/x \in \mathrm{wFilt}_{\mathcal{U}}$, and each C/x is weakly contractible since p is a weak equivalence. The claim follows. $\square$

16.8. **A characterization of distilled ∞-categories.**

16.9. **Corollary.** *A small ∞-category J is distilled iff it is equivalent to the colimit of some functor $f\colon X \to \mathrm{Cat}_\infty$, where $X \in \mathcal{S}$ and f takes values in Filt_ω. Furthermore, X can be taken to be equivalent to the groupoid completion of J.*

16.10. **Corollary.** $\mathrm{Filt}_{\mathtt{pb}}$ *is the smallest filtration class which contains both* Filt_ω *and* $\mathcal{S}$.

Proof. $\mathrm{Filt}_{\mathtt{pb}}$ is stable in Cat_∞ under distilled coliimits. □

16.11. **Corollary.** *If A is an ∞-category with all filtered colimits, and all colimits indexed by small ∞-groupoids, then A has all distilled colimits.*

If $f\colon A \to B$ is a functor between ∞-categories with all distilled coliimits, and if f preserves both filtered colimits and colimits indexed by small ∞-groupoids, then f preserves all distilled colimits.

16.12. *Example.* Let $(P, \leq)$ be a directed set (i.e., a poset which is ω-filtered as an ∞-category), and suppose P is equipped with an action by a discrete group G which acts freely on the underlying set of P. We can define a 1-category J with

- *objects* elements x of P, and
- *morphisms* $x \to y$ the set of $g \in G$ such that $gx \leq y$.

You can show that J is equivalent to the colimit of the evident functor $G \to \mathrm{Cat}_\infty$ defined by the above data. By what we have shown, J is distilled.

16.13. **Filtration classes between distilled and filtered.** We can use this to characterize filtration classes which are "between distilled and ω-filtered".

16.14. **Proposition.** *Let $\mathcal{U}$ be a class of ∞-categories such that $\mathtt{pb} \subseteq \mathcal{U} \subseteq \overline{\omega\text{-}\mathtt{sm}}$. Then J is $\mathcal{U}$-filtered iff*

(1) *J is distilled, and*
(2) *the groupoid completion BJ is $\mathcal{U}$-null.*

Proof. That elements of $\mathrm{Filt}_{\mathcal{U}}$ have property (1) is immediate since $\mathtt{pb} \subseteq \mathcal{U}$, and that they have property (2) follows from (**??**).

Now suppose J is distilled and is such that BJ is $\mathcal{U}$-null. We know (16.9) that J is a colimit in Cat_∞ of some functor $f\colon BJ \to \mathrm{Filt}_\omega \subseteq \mathrm{Cat}_\infty$. Since BJ is $\mathcal{U}$-null it is $\mathcal{U}$-filtered (**??**), and since $\mathrm{Filt}_{\mathcal{U}}$ is stable under $\mathcal{U}$-filtered colimits and contains Filt_ω, we must have (**??**) that the colimit J of f is $\mathcal{U}$-filtered as desired. □

16.15. *Example.* Applying (16.14) to the class $\omega\text{-}\mathtt{smctr}$ of ω-small ∞-categories which are also weakly contractible, we find that $\mathrm{Filt}_{\mathtt{pb}} \approx \mathrm{Filt}_{\omega\text{-}\mathtt{smctr}}$ and thus $\overline{\mathtt{pb}} = \overline{\omega\text{-}\mathtt{smctr}}$.

16.16. *Example.* Recall the doctrine $\omega\text{-}\mathtt{smconn}$ of connected ω-small ∞-categories. We have already seen (5.3) that $\mathrm{Filt}_{\omega\text{-}\mathtt{smconn}}$ is precisely the class of coproducts of ω-filtered ∞-categories. Now (16.14) identifies this with the class of distilled ∞-categories whose groupoid completion is discrete.

16.17. *Example.* Let $\mathcal{U} = \mathtt{pb} \cup \{S^1\}$. Then (16.14) also identifies $\mathrm{Filt}_{\mathcal{U}}$ with the class of distilled ∞-categories with discrete groupoid completion. Thus $\mathrm{Filt}_{\mathcal{U}} = \mathrm{Filt}_{\omega\text{-}\mathtt{smconn}}$ and $\overline{\mathcal{U}} = \overline{\omega\text{-}\mathtt{smconn}}$.

This gives an example of a regular class which is generated by an unsound doctrine $\mathcal{U}$ (6.5), but is also generated by a sound doctrine $\omega\text{-}\mathtt{smconn}$ (5.3).

17. On commuting colimits and limits

I need a to make sure that the various ways of "passing limits across colimits" are equivalent. Not really sure how to do this yet, but I'll put what I have here.

Let J be a small ∞-category, and A any ∞-category. Write $j\colon J \to J^{\triangleright}$ for the inclusion of the right cone, and consider the functors

$$\mathrm{Fun}^{\mathrm{colim}}(J^{\triangleright}, A) \xrightarrow{i} \mathrm{Fun}(J^{\triangleright}, A) \xrightarrow{j^*} \mathrm{Fun}(J, A)$$

where i is the inclusion of the full subcategory of functors $J^{\triangleright} \to A$ which are colimit cones. The composite $j^* \circ i$ is fully faithful **[citation needed]**, and is an equivalence if and only if A has all J-colimits.

17.1. **Lemma.** *For any colimit cone $f\colon J^{\triangleright} \to A$, and any functor $g\colon J^{\triangleright} \to A$, we have that*

$$\mathrm{Map}_{\mathrm{Fun}(J^{\triangleright},A)}(i(f), g) \to \mathrm{Map}_{\mathrm{Fun}(J,A)}(j^*(i(f)), j^*(g))$$

is an equivalence.

Proof. **[citation needed]** □

17.2. **Corollary.** *If A has all J-colimits, then the functor j^* admits a fully faithful left adjoint, which is equivalent to $i \circ (j^* \circ i)^{-1}$.*

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