FROBENIUS PAIRS AND ATIYAH DUALITY

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ABSTRACT. We define a notion of "Frobenius pair", which is a mild generalization of the notion of "Frobenius object" in a monoidal category. We then show that Atiyah duality for smooth manifolds can be encapsulated in the statement that a certain collection of structure obtained from a manifold forms a "commutative Frobenius pair" in the stable homotopy category of spectra.

1. INTRODUCTION

A *Frobenius algebra* over a field k is an associative k-algebra equipped with a k-linear map $\lambda\colon A \to k$ such that the pairing $A \otimes_k A \to k$ defined by $x \otimes y \mapsto \lambda(xy)$ is non-degenerate. One class of Frobenius algebras is produced by *Poincaré duality*: if M is a closed compact manifold which is orientable, the cohomology ring $H^*(M; k)$ admits the structure of a Frobenius algebra. There is a generalization of this notion to an arbitrary monoidal category, which is called a *Frobenius object* [Koc04].

If one wants to refine classical Poincaré duality to an arbitrary generalized cohomology theory, one is lead to *Atiyah duality* [Ati61]. This states that for any closed compact manifold (not necessarily orientable), there exists a non-degenerate pairing

$$M^{-\tau} \wedge M_+ \to S^0$$

in the stable homotopy category; here M_+ denotes the suspension spectrum of M with a disjoint basepoint, $M^{-\tau}$ denotes the Thom spectrum associated to the stable normal bundle of M, and "non-degenerate" means that the adjoint map $M_+ \to \underline{\mathrm{Hom}}(M^{-\tau}, S^0)$ is a weak equivalence, i.e., that M_+ and $M^{-\tau}$ are Spanier-Whitehead dual.

We cannot in general say that M_+ is a Frobenius object in the stable homotopy category; at best this can be done only if M_+ admits a stable framing. By smashing with an appropriate multiplicative cohomology theory, one can obtain a Frobenius object in some category of module spectra; see [Str00] for a treatment along these lines.

The goal of this note is to describe a mild generalization of the notion of a Frobenius object in a monoidal category, which we call a *Frobenius pair*, and to show that Atiyah duality is naturally encapsulated by this definition. Notably, we use this formulation to give a proof of Atiyah duality for smooth manifolds which has a rather formal character, in the sense that we use only the existence of geometric constructions such as the Pontryagin-Thom collapse map, the equivalence of embeddings into Euclidean space of large dimension, and standard formalities about Thom spectra.

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I am sure that the notion which I have called "Frobenius pair" has been encountered elsewhere, though I've been unable to trace it in the literature. I am also sure that the proof of Atiyah duality sketched here is well known; some constructions like the ones described here appear in [Coh04]. The proof given here actually grew out of an attempt to generalize the proof of Poincaré duality in ordinary cohomology described in Chapter 10 of [MS74] to one valid for any cohomology theory. The notion of Frobenius pair and the proof of Atiyah duality taken together make for a nice story, hence this note.

2. Frobenius pairs

The following definition takes place in a monoidal category $\mathfrak{C}$, with unit object $\mathbb{1}$ and associative monoidal product $\otimes\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$. The name "Frobenius pair" is meant to be suggestive of "Frobenius object", except that instead of a single object A with structure, it consists of two objects X and Y, with some additional structure.

Definition of left Frobenius pair. A **left Frobenius pair** in $\mathfrak{C}$ is data $(X, Y, \eta, \mu, \psi, \epsilon, \delta, \phi)$ consisting of

- objects X, Y of $\mathfrak{C}$,
- morphisms

$$
\begin{aligned}
\epsilon &\colon Y \to \mathbb{1}, & \eta &\colon \mathbb{1} \to X, \\
\delta &\colon Y \to Y \otimes Y, & \mu &\colon X \otimes X \to X, \\
\phi &\colon X \to Y \otimes X, & \psi &\colon X \otimes Y \to Y,
\end{aligned}
$$

such that the following hold.

(1a) The triple (X, η, μ) is an associative monoid object (with unit) in $\mathfrak{C}$. That is, the diagrams

$$
\begin{CD}
X \otimes X \otimes X @>{\mu \otimes X}>> X \otimes X \\
@V{X \otimes \mu}VV @VV{\mu}V \\
X \otimes X @>>{\mu}> X
\end{CD}
\qquad
\begin{CD}
\mathbb{1} \otimes X @>{\eta \otimes X}>> X \otimes X @<{X \otimes \eta}<< X \otimes \mathbb{1} \\
@. @VV{\mu}V @. \\
@. X @.
\end{CD}
$$

commute.

(1b) The triple (Y, ϵ, δ) is an associative comonoid object (with counit) in $\mathfrak{C}$. That is, the diagrams

$$
\begin{CD}
Y \otimes Y \otimes Y @<{\delta \otimes Y}<< Y \otimes Y \\
@A{Y \otimes \delta}AA @AA{\delta}A \\
Y \otimes Y @<<{\delta}< Y
\end{CD}
\qquad
\begin{CD}
\mathbb{1} \otimes Y @<{\epsilon \otimes Y}<< Y \otimes Y @>{Y \otimes \epsilon}>> Y \otimes \mathbb{1} \\
@. @AA{\delta}A @. \\
@. Y @.
\end{CD}
$$

commute.

(2a) The pair (Y, ψ) defines a left X-module structure. That is, the diagrams

$$
\begin{array}{ccc}
X \otimes X \otimes Y & \xrightarrow{\mu \otimes Y} & X \otimes Y \\
{\scriptstyle X \otimes \psi} \downarrow & & \downarrow {\scriptstyle \psi} \\
X \otimes Y & \xrightarrow[\psi]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{1} \otimes Y & \xrightarrow{\eta \otimes Y} & X \otimes Y \\
& & \downarrow {\scriptstyle \psi} \\
& & Y
\end{array}
$$

commute.

(2b) The pair (X, ϕ) defines a left Y-comodule structure. That is, the diagrams

$$
\begin{array}{ccc}
Y \otimes Y \otimes X & \xleftarrow{\delta \otimes X} & Y \otimes X \\
{\scriptstyle Y \otimes \phi} \uparrow & & \uparrow {\scriptstyle \phi} \\
Y \otimes X & \xleftarrow[\phi]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{1} \otimes X & \xleftarrow{\epsilon \otimes X} & Y \otimes X \\
& & \uparrow {\scriptstyle \phi} \\
& & X
\end{array}
$$

commute.

(3a) The map ϕ is a homomorphism of left X-modules. That is, the diagram

$$
\begin{array}{ccc}
X \otimes X & \xrightarrow{X \otimes \phi} & X \otimes Y \otimes X \\
{\scriptstyle \mu} \downarrow & & \downarrow {\scriptstyle \psi \otimes X} \\
X & \xrightarrow[\phi]{} & Y \otimes X
\end{array}
$$

commutes.

(3b) The map ψ is a homomorphism of left Y-comodules. That is, the diagram

$$
\begin{array}{ccc}
Y \otimes Y & \xleftarrow{Y \otimes \psi} & Y \otimes X \otimes Y \\
{\scriptstyle \delta} \uparrow & & \uparrow {\scriptstyle \phi \otimes Y} \\
Y & \xleftarrow[\psi]{} & X \otimes Y
\end{array}
$$

commutes.

As a consequence of these axioms, we have the following.

2.1. **Proposition.** *The map ϕ is a homomorphism of right X-modules, and the map ψ is a homomorphism of right Y-comodules. That is, the diagrams*

$$
\begin{array}{ccc}
X \otimes X & \xrightarrow{\phi \otimes X} & Y \otimes X \otimes X \\
{\scriptstyle \mu} \downarrow & & \downarrow {\scriptstyle Y \otimes \mu} \\
X & \xrightarrow[\phi]{} & Y \otimes X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y \otimes Y & \xleftarrow{\psi \otimes Y} & X \otimes Y \otimes Y \\
{\scriptstyle \delta} \uparrow & & \uparrow {\scriptstyle X \otimes \delta} \\
Y & \xleftarrow[\psi]{} & X \otimes Y
\end{array}
$$

commute.

Proof. The left-hand square follows from the commutativity of

$$
\begin{array}{c}
\end{array}
$$

The top diamond commutes by naturality of the monoidal product. The three squares along the middle row commute by (3a), (3b), and (3a) respectively. The lower triangles follow from the unit identities of (1b) and (2b).

The commutativity of the right-hand square is proved similarly. □

Definition of right Frobenius pair. We may similarly define the notion of a **right Frobenius pair** by interchanging "left" and "right" suitably; thus, a right Frobenius pair in $(\mathfrak{C}, \otimes, \mathbb{1})$ is precisely a left Frobenius pair in $(\mathfrak{C}, \otimes^{\mathrm{op}}, \mathbb{1})$, where $A \otimes^{\mathrm{op}} B \stackrel{\text{def}}{=} B \otimes A$.

Definition of commutative Frobenius pair. If $(\mathfrak{C}, \otimes, \mathbb{1})$ is a symmetric monoidal category, we define a **commutative Frobenius pair** in $\mathfrak{C}$ to be a left Frobenius pair such that μ and δ are *commutative* product and coproduct, respectively. A commutative Frobenius pair will necessarily be both a left and right Frobenius pair.

Frobenius objects are Frobenius pairs. Suppose that a left Frobenius pair can be described by

$$(X, Y, \eta, \mu, \psi, \epsilon, \delta, \phi) = (A, A, \eta, \mu, \mu, \epsilon, \delta, \delta).$$

Then we see that $(A, \eta, \mu, \epsilon, \delta)$ is precisely what is usually termed a **Frobenius object** in $\mathfrak{C}$. In particular, axioms (2a) and (2b) are redundant, being in this case consequences of (1a) and (1b). Note that this object will also be a right Frobenius pair.

Frobenius pairs and dualizability. A Frobenius pair always provides a pair of dualizable objects. Thus, if $(X, Y, \eta, \mu, \psi, \epsilon, \delta, \phi)$ is a left Frobenius pair, we may define maps

$$\alpha \stackrel{\text{def}}{=} \phi \circ \eta \colon \mathbb{1} \to Y \otimes X, \qquad \beta \stackrel{\text{def}}{=} \epsilon \circ \psi \colon X \otimes Y \to \mathbb{1},$$

and we have the following proposition.

2.2. **Proposition.** *Given a left Frobenius pair in $\mathfrak{C}$ as above, the data (X, Y, α, β) makes X into a **left dualizable object** of $\mathfrak{C}$. That is, the composites*

$$X = X \otimes \mathbb{1} \xrightarrow{X \otimes \alpha} X \otimes Y \otimes X \xrightarrow{\beta \otimes X} \mathbb{1} \otimes X = X$$

and

$$Y = \mathbb{1} \otimes Y \xrightarrow{\alpha \otimes Y} Y \otimes X \otimes Y \xrightarrow{Y \otimes \beta} Y \otimes \mathbb{1} = Y$$

*are the identity maps of X and Y respectively. Thus, we obtain **duality isomorphisms***

$$\lambda \colon \operatorname{Hom}_{\mathfrak{C}}(X \otimes U, V) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{C}}(U, Y \otimes V),$$

and
$$\rho \colon \mathrm{Hom}_{\mathfrak{C}}(U \otimes Y, V) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}}(U, V \otimes X),$$
natural in objects U and V of $\mathfrak{C}$.

Proof. Straightforward from the definitions. Explicitly,
$$\lambda(f) = (Y \otimes f) \circ (\phi \eta \otimes U), \qquad \lambda^{-1}(g) = (\epsilon \psi \otimes V) \circ (X \otimes g),$$
and
$$\rho(f) = (f \otimes X) \circ (U \otimes \phi \eta), \qquad \rho^{-1}(g) = (V \otimes \epsilon \psi) \circ (g \otimes Y).$$
Note that only properties (3a), (3b), and the unit and counit conditions from (1a), (1b), (2a), (2b) are required; the maps δ and μ are not used in this proof. □

2.3. Corollary. *Given a left Frobenius pair in $\mathfrak{C}$ as above, its duality isomorphisms give correspondences*

$$\mathrm{Hom}_{\mathfrak{C}}(X \otimes X, X) \xleftrightarrow{\ \lambda\ } \mathrm{Hom}_{\mathfrak{C}}(X, Y \otimes X) \xleftrightarrow{\ \rho\ } \mathrm{Hom}_{\mathfrak{C}}(X \otimes Y, Y) \xleftrightarrow{\ \lambda\ } \mathrm{Hom}_{\mathfrak{C}}(Y, Y \otimes Y)$$

$$\mu \longleftrightarrow \phi \longleftrightarrow \psi \longleftrightarrow \delta$$

and
$$\mathrm{Hom}_{\mathfrak{C}}(\mathbb{1}, X) \xleftrightarrow{\ \rho\ } \mathrm{Hom}_{\mathfrak{C}}(Y, \mathbb{1})$$

$$\eta \longleftrightarrow \epsilon$$

Proof. Straightforward, using the axioms and (2.2). □

2.4. Remark. If $\mathfrak{C}$ is a *symmetric closed monoidal category*, and so admits an internal function object $\underline{\mathrm{Hom}}(-,-)$ satisfying $\mathrm{Hom}_{\mathfrak{C}}(Z \otimes A, B) \approx \mathrm{Hom}_{\mathfrak{C}}(A, \underline{\mathrm{Hom}}(Z, B))$, then dualizability of X gives rise to isomorphisms
$$\underline{\mathrm{Hom}}(X \otimes U, V) \approx \underline{\mathrm{Hom}}(U, Y \otimes V).$$
In particular, by taking $U = V = \mathbb{1}$ we obtain an isomorphism $\underline{\mathrm{Hom}}(X, \mathbb{1}) \approx Y$.

3. Atiyah duality

In this section, we associate to a compact smooth manifold M a Frobenius pair in the stable homotopy category with objects $\Sigma_+^\infty M$ and $M^{-\tau}$. We do this simply by constructing the relevant objects and maps, and showing that the appropriate diagrams commute up to homotopy.

We will use the following constructions and notation.
- For a vector bundle $V \to M$, we write M^V for the associated Thom space; we may regard M^V as a spectrum by applying Σ^∞.
- In particular, $M^0 \approx \Sigma^\infty(M_+)$, and $(*)^0$ is the stable 0-sphere, which we denote by $\mathbb{1}$.
- Given an embedding $f \colon M \hookrightarrow N$ of manifolds, we write $\nu(f) \to M$ for the normal bundle of the embedding.
- For an embedding $f \colon M \hookrightarrow N$, we write $\widehat{f} \colon N^0 \to M^{\nu(f)}$ for the associated collapse map.

- More generally, given a sequence of embeddings $f\colon M \hookrightarrow N$ and $g\colon N \hookrightarrow P$, we obtain a collapse map of N to M with respect to the ambient space P, of the form
$$\widehat{f}^{\nu(g)}\colon N^{\nu(g)} \to M^{\nu(gf)}.$$

- Given embedding $f\colon M \hookrightarrow N$ and a vector bundle $V \to N$, we obtain a collapse map $\widehat{f}^{V}\colon N^V \to M^{f^*V \oplus \nu(f)}$ by regarding $N \subset V$ as the 0-section, so that $f^*V \oplus \nu(f)$ is equivalent to the normal bundle of $M \subset V$; we may think of the map $\widehat{f}^V$ as a collapse map "twisted" by the bundle V.

- We extend our Thom space notation to *Thom spectra of virtual bundles*. Thus, M^{V-W} denotes the Thom spectrum of the virtual bundle $V - W$. Likewise, given any smooth map $f\colon M \to N$ and virtual bundle $V \to N$, we can associate maps $f^V\colon M^{f^*V} \to N^V$ and $\widehat{f}^V\colon N^V \to M^{\nu(f)\oplus V}$.

Especially significant is the following observation.

Homotopy invariance of the collapse map. The construction of collapse maps is *homotopy invariant* with respect to smooth isotopies. That is, given smooth isotopies $f_0 \sim f_1\colon M \hookrightarrow N$ and $g_0 \sim g_1\colon N \hookrightarrow P$ we obtain homotopy equivalences $N^{\nu(g_0)} \approx N^{\nu(g_1)}$ and $M^{\nu(g_0 f_0)} \approx M^{\nu(g_1 f_1)}$, which fit in a homotopy commutative diagram

$$
\begin{array}{ccc}
M^{\nu(g_0 f_0)} & \xleftarrow{\;\widehat{f_0}^{\nu(g_0)}\;} & N^{\nu(g_0)} \\
{\scriptstyle\sim}\big\updownarrow & & \big\updownarrow{\scriptstyle\sim} \\
M^{\nu(g_1 f_1)} & \xleftarrow[\;\widehat{f_1}^{\nu(g_1)}\;]{} & N^{\nu(g_1)}
\end{array}
$$

We will usually apply this in cases where the *space of choices* of suitable isotopies turns out to be contractible (or at least, with connectivity approaching ∞ as some parameter $B \to \infty$). When we can restrict to such a space of choices for our isotopies, it is clear that all such choices lead to the *same* (up to homotopy!) homotopy equivalences $N^{\nu(g_0)} \approx N^{\nu(g_1)}$ and $M^{\nu(g_0 f_0)} \approx M^{\nu(g_1 f_1)}$, and thus we may unambiguously identify these objects (in the homotopy category of spaces, or spectra as may be); with respect to this identification, the collapse maps $\widehat{f_0}^{\nu(g_0)}$ and $\widehat{f_1}^{\nu(g_1)}$ are the same up to homotopy.

The Frobenius pair structure on the stabilization of a smooth manifold. The Frobenius pair associated to a smooth compact manifold M consists of spectra $X = M^{-\tau}$ and $Y = M^0$, and maps

$$
\begin{aligned}
\epsilon &\colon M^0 \to \mathbb{1}, & \eta &\colon \mathbb{1} \to M^{-\tau}, \\
\delta &\colon M^0 \to M^0 \wedge M^0, & \mu &\colon M^{-\tau} \wedge M^{-\tau} \to M^{-\tau}, \\
\phi &\colon M^{-\tau} \to M^0 \wedge M^{-\tau}, & \psi &\colon M^{-\tau} \wedge M^0 \to M^0.
\end{aligned}
$$

This data will satisfy the axioms for a Frobenius pair in the homotopy category of spectra. In brief, the maps in the left-hand column (ϵ, δ, and ϕ) are stabilizations of certain maps between manifolds (or bundles), while the maps in the right-hand column (η, μ, and ψ) are obtained as Pontryagin-Thom collapse maps associated to certain embeddings.

We take up the definition of each map, and a sketch of the proofs of each axiom, in turn.

The maps ϵ and δ. These are just the maps $p^0 \colon M^0 \to (*)^0 \approx \mathbb{1}$ and $d^0 \colon M^0 \to (M \times M)^0 \approx M^0 \wedge M^0$ of spectra obtained from the projection and diagonal maps $p \colon M \to *$ and $d \colon M \to M \times M$

Axiom (1b). Clear.

The map η. Choose an embedding $j \colon M \hookrightarrow \mathbb{R}^B$ into a Euclidean space of large dimension. We obtain the Pontryagin-Thom collapse map

$$\widehat{j} \colon S^B \approx (\mathbb{R}^B)^0 \to M^{\nu(j)}.$$

For B sufficiently large, any two embeddings are isotopic; in fact, as $B \to \infty$, the space of embeddings becomes contractible. Thus, this construction produces a well-defined stable homotopy class of maps

$$\eta = \widehat{p} \colon \mathbb{1} \to M^{-\tau};$$

we write $-\tau$ for the virtual bundle $\nu(j) - \underline{B}$.

The map ϕ. Choose an embedding $j \colon M \hookrightarrow \mathbb{R}^B$, and thus a normal bundle $\nu(j)$ over M. Pulling back the bundle $0 \times \nu(j)$ over $M \times M$ along the diagonal map induces a map

$$d^{0 \times \nu(j)} \colon M^{\nu(j)} \to (M \times M)^{0 \times \nu(j)}$$

on Thom spaces. Stabilizing, we obtain $\phi = d^{0 \times (-\tau)} \colon M^{-\tau} \to M^0 \wedge M^{-\tau}$.

Axiom (2b). This is standard.

The map μ. Pick any pair of embeddings $j_1, j_2 \colon M \hookrightarrow \mathbb{R}^B$, and consider the sequence of embeddings

$$M \xrightarrow{d} M \times M \xrightarrow{j_1 \times j_2} \mathbb{R}^B \times \mathbb{R}^B.$$

We obtain a collapse map associated to the embedding d inside the ambient space $\mathbb{R}^B \times \mathbb{R}^B$, of the form

$$\widehat{d}^{\nu(j_1) \times \nu(j_2)} \colon (M \times M)^{\nu(j_1) \times \nu(j_2)} \to M^{\nu((j_1, j_2))}.$$

As noted, all choices of embedding of $M \hookrightarrow \mathbb{R}^\infty$ live in a contractible parameter space; thus after stabilizing, the embeddings j_1, j_2, and (j_1, j_2) give the same Thom spectrum $M^{-\tau}$, and we obtain a map $\mu = \widehat{d}^{(-\tau) \times (-\tau)} \colon M^{-\tau} \wedge M^{-\tau} \approx (M \times M)^{(-\tau) \times (-\tau)} \to M^{-\tau}$. (Compare [Coh04], where the map μ is constructed and refined to a strictly commutative multiplication on the spectrum (without strict unit).)

Axiom (1a). To prove the unit identity $\mu \circ (\eta \otimes X) = 1_X$, consider the sequence of embeddings

$$M \xrightarrow{d} M \times M \xrightarrow{j_1 \times \mathrm{id}} \mathbb{R}^B \times M \xrightarrow{\mathrm{id} \times j_2} \mathbb{R}^B \times \mathbb{R}^B.$$

The collapse map obtained from the embedding $(j_1, \mathrm{id}) = (j_1 \times \mathrm{id}) \circ d \colon M \hookrightarrow \mathbb{R}^B \times M$ inside the ambient space $\mathbb{R}^B \times \mathbb{R}^B$ is a composite

$$S^B \wedge M^{\nu(j_2)} \to (M \times M)^{\nu(j_1) \times \nu(j_2)} \to M^{\nu(j_1, j_2)},$$

which can be written as

$$S^B \wedge M^{\nu(j_2)} \xrightarrow{\widehat{j_1} \wedge \mathrm{id}} M^{\nu(j_1)} \wedge M^{\nu(j_2)} \xrightarrow{\widehat{d}^{\nu(j_1) \times \nu(j_2)}} M^{\nu(j_1, j_2)},$$

which when stabilized realizes $\mathbb{1} \wedge M^{-\tau} \xrightarrow{\eta \wedge \mathrm{id}} M^{-\tau} \wedge M^{-\tau} \xrightarrow{\mu} M^{-\tau}$.

On the other hand, the map $j_1 \colon M \to \mathbb{R}^B$ is homotopic through smooth maps to a constant map $0 \colon M \to \mathbb{R}^B$, and such a homotopy produces an isotopy $(j_1, \mathrm{id}) \sim (0, \mathrm{id})$ of embeddings $M \to \mathbb{R}^B \times M$. (The space of smooth maps $j_1 \colon M \to \mathbb{R}^B$ is contractible, so there is no ambiguity created by the choice of homotopy.) The sequence of embeddings

$$M \xrightarrow{(0, \mathrm{id})} \mathbb{R}^B \times M \xrightarrow{\mathrm{id} \times j_2} \mathbb{R}^B \times \mathbb{R}^B$$

induces a collapse map

$$S^B \wedge M^{\nu(j_2)} \to M^{\nu(0, j_2)} \approx M^{\underline{B} \oplus \nu(j_2)}$$

which is manifestly homotopic to the identity. According to the homotopy invariance of the collapse map with respect to isotopies, this map may be naturally identified up to homotopy with the composite $\widehat{d}^{\nu(j_1) \times \nu(j_2)} \circ (\widehat{j_1} \wedge \mathrm{id})$.

The unit identity $\mu \circ (X \otimes \eta) = 1_X$ is proved similarly.

To prove the associativity identity, choose embeddings $j_1, j_2, j_3 \colon M \hookrightarrow \mathbb{R}^B$, and consider the commutative square of embeddings

$$
\begin{array}{ccc}
M & \xrightarrow{\ \ d\ \ } & M \times M \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d \times \mathrm{id}} \\
M \times M & \xrightarrow[\mathrm{id} \times d]{} & M \times M \times M
\end{array}
$$

The diagram of associated collapse maps of the submanifolds inside $\mathbb{R}^B \times \mathbb{R}^B \times \mathbb{R}^B$ has the form

$$
\begin{array}{ccc}
M^{\nu((j_1, j_2, j_3))} & \xleftarrow{\ \widehat{d}^{\nu((j_1, j_2)) \times \nu(j_3)}\ } & (M \times M)^{\nu((j_1, j_2)) \times \nu(j_3)} \\
{\scriptstyle \widehat{d}^{\nu(j_1) \times \nu((j_2, j_3))}}\uparrow & & \uparrow{\scriptstyle \widehat{d \times \mathrm{id}}^{\nu(j_1) \times \nu(j_2) \times \nu(j_3)}} \\
(M \times M)^{\nu(j_1) \times \nu((j_2, j_3))} & \xleftarrow[\widehat{\mathrm{id} \times d}^{\nu(j_1) \times \nu(j_2) \times \nu(j_3)}]{} & (M \times M \times M)^{\nu(j_1) \times \nu(j_2) \times \nu(j_3)}
\end{array}
$$

which after stabilizing is the desired commutative diagram.

The map ψ. Pick an embedding $j \colon M \hookrightarrow \mathbb{R}^B$, and consider the sequence of embeddings

$$M \xrightarrow{d} M \times M \xrightarrow{j \times \mathrm{id}} \mathbb{R}^B \times M.$$

Forming the collapse map of d with respect to the ambient space $\mathbb{R}^B \times M$ gives

$$\widehat{d}^{\nu(j) \times 0} \colon (M \times M)^{\nu(j) \times 0} \to M^{\nu((j, \mathrm{id}))}.$$

By choosing any smooth homotopy of $j \colon M \to \mathbb{R}^B$ to a constant map, we obtain an isotopy $(j, \mathrm{id}) \sim (0, \mathrm{id})$, which provides a bundle equivalence $\nu((j, \mathrm{id})) \approx \nu((0, \mathrm{id})) = \underline{B}$. That is, we obtain a map $M^{\nu(j)} \wedge M^0 \to M^{\underline{B}}$; we let $\psi = \widehat{d}^{(-\tau) \times 0} \colon M^{-\tau} \wedge M^0 \to M^0$ be the map obtained after stabilization.

Axiom (2a). Choose an embedding $j\colon M \hookrightarrow \mathbb{R}^B$. Since the composite

$$M \xrightarrow{d} M \times M \xrightarrow{j \times \mathrm{id}} \mathbb{R}^B \times M$$

is isotopic to $(0, \mathrm{id})\colon M \hookrightarrow \mathbb{R}^B \times M$ by means of a smooth homotopy $j \sim 0$, the composite of collapse maps

$$S^B \wedge M \xrightarrow{\widehat{j} \wedge \mathrm{id}} M^{\nu(j)} \wedge M^0 \xrightarrow{\widehat{d}^{\nu(j) \times 0}} M^{\nu(j,\mathrm{id})}$$

is homotopic to the collapse map of $(0, \mathrm{id})\colon M \to \mathbb{R}^B \times M$, which is homotopic to the identity map of $S^B \wedge M^0$; this proves the unit identity.

Likewise, the commutative square of diagonal embeddings into $M \times M \times M \subset \mathbb{R}^B \times \mathbb{R}^B \times M$ induces a commutative square of collapse maps (relative to the ambient space $\mathbb{R}^B \times \mathbb{R}^B \times M$),

$$
\begin{array}{ccc}
M^{\nu((j_1,j_2,\mathrm{id}))} & \xleftarrow{\widehat{d}^{\nu(j_1,j_2) \times 0}} & (M \times M)^{\nu(j_1,j_2) \times 0} \\
{\scriptstyle \widehat{d}^{\nu(j_1) \times \nu((j_2,\mathrm{id}))}} \uparrow & & \uparrow {\scriptstyle \widehat{d \times \mathrm{id}}^{\nu(j_1) \times \nu(j_2) \times 0}} \\
(M \times M)^{\nu(j_1) \times \nu((j_2,\mathrm{id}))} & \xleftarrow{\widehat{\mathrm{id} \times d}^{\nu(j_1) \times \nu(j_2) \times 0}} & (M \times M \times M)^{\nu(j_1) \times \nu((j_2) \times 0}
\end{array}
$$

which is precisely the associativity identity.

A transversality diagram. Suppose given a commutative diagram of manifolds

$$
\begin{array}{ccccc}
A_1 & \xrightarrow{h_1} & B_1 & \xrightarrow{k_1} & C_1 \\
{\scriptstyle f} \downarrow & & {\scriptstyle g} \downarrow & & \\
A_2 & \xrightarrow{h_2} & B_2 & \xrightarrow{k_2} & C_2
\end{array}
$$

in which h_1, h_2, k_1, k_2 are embeddings, and A_1 is the transversal intersection of A_2 along g. Furthermore, suppose we are given a bundle equivalence $\beta\colon \nu(k_1) \xrightarrow{\sim} g^*\nu(k_2)$ over B_1, which induces a bundle equivalence $\alpha\colon \nu(k_1 h_1) \xrightarrow{\sim} f^*\nu(k_2 h_2)$ over A_1 (using the evident equivalence $\nu(h_1) \approx f^*\nu(h_2)$).

Then we obtain a homotopy commutative diagram

$$
\begin{array}{ccc}
A_1^{\nu(k_1 h_1)} & \xleftarrow{\widehat{h_1}^{\nu(k_1)}} & B_1^{\nu(k_1)} \\
{\scriptstyle f^{\nu(k_2 h_2)}} \downarrow & & \downarrow {\scriptstyle g^{\nu(k_2)}} \\
A_2^{\nu(k_2 h_2)} & \xleftarrow{\widehat{h_2}^{\nu(k_2)}} & B_2^{\nu(k_2)}
\end{array}
$$

in which the vertical maps are inclusions of Thom spaces induced by the pullback squares

$$
\begin{array}{ccc}
\nu(k_1 h_1) & \xrightarrow{\alpha} & \nu(k_2 h_2) \\
\downarrow & & \downarrow \\
A_1 & \xrightarrow{f} & A_2
\end{array}
\qquad
\begin{array}{ccc}
\nu(k_1) & \xrightarrow{\beta} & \nu(k_2) \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{g} & B_2
\end{array}
$$

and the horizontal maps are collapse maps.

Axiom (3a). Consider

$$M \xrightarrow{d} M \times M \xrightarrow{j_1 \times j_3} \mathbb{R}^B \times \mathbb{R}^B$$

$$\downarrow d \qquad\qquad \downarrow \mathrm{id} \times d$$

$$M \times M \xrightarrow{d \times \mathrm{id}} M \times M \times M \xrightarrow{j_1 \times \mathrm{id} \times j_3} \mathbb{R}^B \times M \times \mathbb{R}^B$$

where $j_1, j_3 \colon M \hookrightarrow \mathbb{R}^B$ are embeddings, where

$$\beta \colon \nu(j_1) \times \nu(j_3) \xrightarrow{\sim} (\mathrm{id} \times d)^*(\nu(j_1) \times 0 \times \nu(j_3))$$

is the obvious bundle isomorphism, and

$$\alpha \colon \nu((j_1, j_3)) \xrightarrow{\sim} d^*(\nu((j_1, \mathrm{id})) \times \nu(j_3))$$

is the bundle isomorphism obtained by pulling back β along the horizontal maps. We obtain a commutative square

$$M^{\nu((j_1,j_3))} \xleftarrow{\widehat{d}^{\nu(j_1) \times \nu(j_3)}} (M \times M)^{\nu(j_1) \times \nu(j_3)}$$

$$\downarrow{d^{\nu((j_1,\mathrm{id})) \times \nu(j_3)}} \qquad\qquad \downarrow{(\mathrm{id} \times d)^{\nu(j_1) \times 0 \times \nu(j_3)}}$$

$$(M \times M)^{\nu((j_1,\mathrm{id})) \times \nu(j_3)} \xleftarrow{\widehat{d \times \mathrm{id}}^{\nu(j_1) \times 0 \times \nu(j_3)}} (M \times M \times M)^{\nu(j_1) \times 0 \times \nu(j_3)}$$

Choose a smooth homotopy of $j_1 \colon M \to \mathbb{R}^B$ to a constant map, thus producing isotopies $(j_1, j_3) \sim (0, j_3)$ of embeddings $M \hookrightarrow \mathbb{R}^B \times \mathbb{R}^B$ and $(j_1, \mathrm{id}) \times j_3 \sim (0, \mathrm{id}) \times j_3$ of embeddings $M \times M \hookrightarrow \mathbb{R}^B \times M \times \mathbb{R}^B$. In addition, we may use this same homotopy to form a 1-parameter family of bundle maps over $d \colon M \to M \times M$, between α and the evident bundle isomorphism

$$\alpha' \colon \nu((0, j_3)) \xrightarrow{\sim} d^*(\nu((0, \mathrm{id})) \times \nu(j_3)).$$

Thus, $d^{\nu((j_1,\mathrm{id})) \times \nu(j_3)} \colon M^{\nu((j_1,j_3))} \to (M \times M)^{\nu((j_1,\mathrm{id})) \times \nu(j_3)}$ is homotopic to

$$M^{\underline{B} \oplus \nu(j_3)} \to M^{\underline{B}} \wedge M^{\nu(j_3)}.$$

After stabilizing, the above diagram is the homotopy commutative diagram

$$M^{-\tau} \xleftarrow{\mu} M^{-\tau} \wedge M^{-\tau}$$

$$\downarrow{\phi} \qquad\qquad \downarrow{\mathrm{id} \wedge \phi}$$

$$M^0 \wedge M^{-\tau} \xleftarrow{\psi \wedge \mathrm{id}} M^{-\tau} \wedge M^0 \wedge M^{-\tau}$$

Axiom (3b). Consider

$$M \xrightarrow{d} M \times M \xrightarrow{j \times \mathrm{id}} \mathbb{R}^B \times M$$

$$\downarrow d \qquad\qquad \downarrow d \times \mathrm{id}$$

$$M \times M \xrightarrow{\mathrm{id} \times d} M \times M \times M \xrightarrow{\mathrm{id} \times j \times \mathrm{id}} M \times \mathbb{R}^B \times M$$

where $j\colon M \hookrightarrow \mathbb{R}^B$ is an embedding, where

$$\beta\colon \nu(j) \times 0 \xrightarrow{\sim} (d \times \mathrm{id})^*(0 \times \nu(j) \times 0)$$

is the obvious bundle isomorphism, and

$$\alpha\colon \nu((j,\mathrm{id})) \xrightarrow{\sim} d^*(0 \times \nu(j,\mathrm{id}))$$

is the bundle isomorphism obtained by pulling back β along the horizontal maps. We obtain a commutative square

$$\begin{array}{ccc}
M^{\nu((j,\mathrm{id}))} & \xleftarrow{\widehat{d}^{\nu(j)\times 0}} & (M \times M)^{\nu(j)\times 0} \\
{\scriptstyle d^{0\times\nu((j,\mathrm{id}))}}\downarrow & & \downarrow{\scriptstyle (d\times\mathrm{id})^{0\times\nu(j)\times 0}} \\
(M \times M)^{0\times\nu((j,\mathrm{id}))} & \xleftarrow{\widehat{\mathrm{id}\times d}^{0\times\nu(j)\times 0}} & (M \times M \times M)^{0\times\nu(j)\times 0}
\end{array}$$

Choose a smooth homotopy of $j\colon M \to \mathbb{R}^B$ to a constant map, thus producing isotopies $(j,\mathrm{id}) \sim (0,\mathrm{id})$ of embeddings $M \hookrightarrow \mathbb{R}^B \times M$ and $\mathrm{id}\times(j,\mathrm{id}) \sim \mathrm{id}\times(0,\mathrm{id})$ of embeddings $M \times M \hookrightarrow M \times \mathbb{R}^B \times M$. In addition, we may use this same homotopy to form a 1-parameter family of bundle maps over $d\colon M \to M \times M$, between α and the evident bundle isomorphism

$$\alpha'\colon \underline{B} \xrightarrow{\sim} d^*(0 \times \underline{B}).$$

Thus, $d^{0\times\nu((j,\mathrm{id}))}\colon M^{\nu((j,\mathrm{id}))} \to (M \times M)^{0\times\nu((j_1,\mathrm{id}))}$ is homotopic to

$$M^{\underline{B}} \to M^0 \wedge M^{\underline{B}}.$$

After stabilizing, the above diagram is the homotopy commutative diagram

$$\begin{array}{ccc}
M^0 & \xleftarrow{\psi} & M^{-\tau} \wedge M^0 \\
{\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \phi\wedge\mathrm{id}} \\
M^0 \wedge M^0 & \xleftarrow[\mathrm{id}\wedge\psi]{} & M^0 \wedge M^{-\tau} \wedge M^0
\end{array}$$

Commutativity. Let $\sigma\colon X \wedge Y \to Y \wedge X$ denote the symmetry of the smash product in the homotopy category of spectra. It is immediate that $\sigma \circ \delta \approx \delta$, from the symmetry of the diagonal embedding $d\colon M \to M \times M$. To show that $\mu \circ \sigma \approx \mu$, it suffices to note that if $j_1, j_2\colon M \hookrightarrow \mathbb{R}^B$ are embeddings, then for B sufficiently large the embeddings $(j_1,j_2), (j_1,j_1)\colon M \hookrightarrow \mathbb{R}^B \times \mathbb{R}^B$ are isotopic,

Atiyah duality. We have shown the following.

3.1. **Proposition.** *Let M be a smooth compact manifold. Then the pair of spectra $\Sigma^\infty M_+$ and $M^{-\tau}$ admit the structure of a commutative Frobenius pair in the homotopy category of spectra.*

As a consequence of general properties of Frobenius pairs, we recover Atiyah duality.

3.2. **Corollary** (Atiyah)**.** *There is a weak equivalence between $\Sigma^\infty M_+$ and the Spanier-Whitehead dual of $M^{-\tau}$.*

3.3. *Remark.* We can extend the above arguments to deal with duality for manifolds with boundary. Thus, if M is a smooth compact manifold with boundary $\partial M = N$, then can we take

$$X = (M/N)^{-\tau}, \qquad Y = M^0,$$

and define maps

$$\epsilon \colon M^0 \to \mathbb{1}, \qquad\qquad\qquad \eta \colon \mathbb{1} \to (M/N)^{-\tau},$$
$$\delta \colon M^0 \to M^0 \wedge M^0, \qquad\qquad \mu \colon (M/N)^{-\tau} \wedge (M/N)^{-\tau} \to (M/N)^{-\tau},$$
$$\phi \colon (M/N)^{-\tau} \to M^0 \wedge (M/N)^{-\tau}, \qquad \psi \colon (M/N)^{-\tau} \wedge M^0 \to M^0,$$

which define a Frobenius pair.

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