Power operations in Morava E-theory

Charles Rezk

University of Illinois at Urbana-Champaign

March 24, 2010

http://www.math.uiuc.edu/~rezk/baltimore-2010-power-ops-handout.pdf

Part I: What are power operations?

$h^* =$ multiplicative cohomology theory: $h^p(X) \otimes h^q(X) \to h^{p+q}(X)$.
m-th power map:

$$x \mapsto x^m \colon h^q(X) \to h^{mq}(X).$$

If h comes from a structured commutative ring spectrum, refine m-th power map to P^m:

$$\begin{array}{ccc} & & h^0(X \times B\Sigma_m) \\ & \overset{P^m}{\nearrow} & \Big\downarrow {\scriptstyle [* \to B\Sigma_m]} \\ h^0(X) & \xrightarrow[x \mapsto x^m]{} & h^0(X) \end{array}$$

- P_m is multiplicative, not additive.
- Pairing with $\alpha \in h_0(B\Sigma_m)$ gives an operation $Q_\alpha \colon h^0(X) \to h^0(X)$.
- Q_α is *additive* iff $\alpha \in$ Primitives of $\bigoplus_m h_0(B\Sigma_m)$.

$R =$ commutative S-algebra.
$M =$ an R-module. Note: $[R, M]_R \approx [S, M]_S \approx \pi_0 M$.
Free commutative R-algebra on M:

$$\mathbb{P}_R M = \bigvee_{m \geq 0} \mathbb{P}_R^m M \approx \bigvee_{m \geq 0} (\underbrace{M \wedge_R \cdots \wedge_R M}_{m \text{ times}})_{h\Sigma_m}$$

commutative R-algebra $A =$ algebra for the monad $\mathbb{P}_R$, determined by

$$\mu \colon \mathbb{P}_R A \to A.$$

$A =$ commutative R-algebra.

- Choose $\alpha\colon S \to \mathbb{P}_R^m(R) \approx R \wedge B\Sigma_m^+$ (map of spectra).
- Represent $x \in \pi_0 A$ by $f_x\colon R \to A$.
- $$\mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f_x)} \mathbb{P}_R^m(A)$$

Remarks:

- $Q_\alpha\colon \pi_0 A \to \pi_0 A$ may not be additive or multiplicative.
- Can get $Q_\alpha\colon \pi_q A \to \pi_{q+r} A$ from

$$\alpha\colon \Sigma^{q+r} R \to \mathbb{P}_R^m(\Sigma^q R) \approx R \wedge B\Sigma_m^{qV_m}.$$

Part I: Deformations & Morava E-theory

Let G_0 = height n formal group over perfect field k, $\operatorname{char} k = p$, $n < \infty$.
Let R = complete local ring, $\pi\colon R \to R/\mathfrak{m}$.

Definition

A **deformation** of G_0 to R is (G, i, ψ):

- G a formal group over R,
- $i\colon k \to R/\mathfrak{m}$,
- $\psi\colon \pi^* G \xrightarrow{\sim} i^* G_0$ iso of formal groups over $R/\mathfrak{m}$.

Theorem (Lubin-Tate)

There is a universal example of a deformation of G_0, defined over $E_0 \approx \mathbb{W}_p k[[u_1, \ldots, u_{n-1}]]$.

Theorem (Morava; Hopkins-Miller)

Given G_0/k, there is a corresponding even periodic commutative S-algebra $E = E_{G_0/k}$, whose formal group is the universal deformation of G_0.

Example 1: p-complete K-algebras [McClure]

K = complex K-theory spectrum.

p-**complete** K-**algebra**: commutative K-algebra A such that $A \approx A^\wedge_p$.

$K^\wedge_p$ is associated to universal deformation of $\widehat{G}_m$ (height 1).

Operations on π_0 of p-complete K-algebra (θ-ring)

$\psi^p\colon \pi_0 A \to \pi_0 A$ such that

- $\psi^p(x+y) = \psi^p(x) + \psi^p(y)$.
- $\psi^p(1) = 1$.
- $\psi^p(xy) = \psi^p(x)\psi^p(y)$.
- $\psi^p(x) \equiv x^p \mod p$. $\theta\colon \pi_0 A \to \pi_0 A$ such that $\psi^p(x) = x^p + p\,\theta(x)$.

ψ^p and θ correspond to elements of $\alpha \in K^\wedge_0 B\Sigma_p$.

$$K^\wedge_q X \stackrel{\text{def}}{=} \pi_q\left((K \wedge X)^\wedge_p\right).$$

ψ^p is the pth **Adams operation**.

Example 2: Morava E-theory ($n = 2, p = 2$)

- $C_0/\mathbb{F}_2$ = supersingular elliptic curve.
- $\widehat{C}_0$ = formal completion — formal group of height 2.
- E = Landweber exact spectrum associated to universal deformation of $\widehat{C}$.

$$\pi_* E \approx \mathbb{Z}_2[\![a]\!][u, u^{-1}], \qquad |a| = 0, |u| = 2.$$

  Note: $K(2)$ is $E/(2, a)$ (sort of).
- E is a commutative S-algebra (Hopkins-Miller Theorem).

Next slide: calculation of the algebraic structure of power operations for $K(2)$-local commutative E-algebras (R., prefigured by Kashiwabara 1995).

Example 2 (continued): Formulas

$A = K(2)$-local commutative E-algebra ($\pi_0 A$ is an $E_0 = \mathbb{Z}_2[\![a]\!]$-algebra).

Operations on π_0 of $K(2)$-local E-algebra

$Q_0, Q_1, Q_2 \colon \pi_0 A \to \pi_0 A$ such that

- $Q_i(x+y) = Q_i(x) + Q_i(y)$
- $Q_0(ax) = a^2\, Q_0(x) - 2a\, Q_1(x) + 6\, Q_2(x)$
  $Q_1(ax) = 3\, Q_0(x) + a\, Q_2(x)$
  $Q_2(ax) = -a\, Q_0(x) + 3\, Q_1(x)$
- $Q_1 Q_0(x) = 2\, Q_2 Q_1(x) - 2\, Q_0 Q_2(x)$
  $Q_2 Q_0(x) = Q_0 Q_1(x) + a\, Q_0 Q_2(x) - 2\, Q_1 Q_2(x)$
- $Q_0(1) = 1,\ Q_1(1) = Q_2(1) = 0$
- $Q_0(xy) = Q_0 x Q_0 y + 2 Q_1 x Q_2 y + 2 Q_2 x Q_1 y$
  $Q_1(xy) = Q_0 x Q_1 y + Q_1 x Q_0 y + a Q_1 x Q_2 y + a Q_2 x Q_1 y + 2 Q_2 x Q_2 y$
  $Q_2(xy) = Q_0 x Q_2 y + Q_2 x Q_0 y + Q_1 x Q_1 y + a Q_2 x Q_2 y$
- $Q_0(x) \equiv x^2 \mod 2$ $\theta \colon \pi_0 A \to \pi_0 A$ such that $Q_0(x) = x^2 + 2\,\theta(x)$

The ring Γ of power operations

Associative ring containing $E_0 = \mathbb{Z}_2[\![a]\!]$ and generators Q_0, Q_1, Q_2, and subject to relations

$$Q_0\, a = a^2\, Q_0 - 2a\, Q_1 + 6\, Q_2$$
$$Q_1\, a = 3\, Q_0 + a\, Q_2$$
$$Q_2\, a = -a\, Q_0 + 3\, Q_1$$

$$Q_1 Q_0 = 2\, Q_2 Q_1 - 2\, Q_0 Q_2$$
$$Q_2 Q_0 = Q_0 Q_1 + a\, Q_0 Q_2 - 2\, Q_1 Q_2$$

Γ has "admissible basis" as left $\mathbb{Z}_2[\![a]\!]$ module:

$$Q_0^i Q_{j_1} \cdots Q_{j_r}, \qquad i \geq 0,\ j_k \in \{1, 2\}$$

Kashiwabara (1995): gives admissible basis for $\bar{\Gamma} = \mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![a]\!]} \Gamma$.
Problem: $\bar{\Gamma}$ is not a ring! (Kashiwabara knows this.)
He describes ring structure modulo indeterminacy.

Example 2 (continued): Coproduct on Γ

"Cartan formula" is encoded by a coproduct.

Cocommutative coalgebra structure on Γ

$\epsilon\colon \Gamma \to E_0$ and $\Delta\colon \Gamma \to {}_{E_0}\Gamma \otimes {}_{E_0}\Gamma$ by

$$\epsilon(Q_0) = 1, \qquad \epsilon(Q_1) = 0 = \epsilon(Q_2)$$

$$\begin{aligned}
\Delta(Q_0) &= Q_0 \otimes Q_0 + 2Q_1 \otimes Q_2 + 2Q_2 \otimes Q_1 \\
\Delta(Q_1) &= Q_0 \otimes Q_1 + Q_1 \otimes Q_0 + aQ_1 \otimes Q_2 + aQ_2 \otimes Q_1 + 2Q_2 \otimes Q_2 \\
\Delta(Q_2) &= Q_0 \otimes Q_2 + Q_2 \otimes Q_0 + Q_1 \otimes Q_1 + aQ_2 \otimes Q_2
\end{aligned}$$

(${}_{E_0}M \otimes {}_{E_0}N$ means tensor using left-module structures.)
Coproduct and product "commute".

Conclusion

Γ is a **twisted bialgebra** over E_0 (like a Hopf algebra, but E_0 isn't central).
Left Γ-modules have a symmetric monoidal tensor product: $M \otimes_{E_0} N$.

Example 2: Summary

Definition

A Γ-**ring** is a commutative ring object in Γ-modules.

Definition

An **amplified** Γ-**ring** is a Γ-ring B equipped with $\theta\colon B \to B$ such that $Q_0(x) = x^2 + 2\,\theta(x)$ (together with formulas for $\theta(x+y)$, $\theta(xy)$, $\theta(ax)$).

In summary:

Proposition

For A a $K(2)$-local commutative E-algebra, $\pi_0 A$ naturally has the structure of an amplified Γ-ring.
$\pi_0 L_{K(2)} \mathbb{P}_E(E) \approx F^{\wedge}_{(2,a)}$, with $F =$ free amplified Γ-ring on one generator.

This can be extended to non-zero degrees:
$\pi_* A$ is a **graded amplified** Γ-**ring**, etc.

Part I: The general pattern

This is the general pattern for any Morava E-theory spectrum.

Power operations for Morava E-theory (height n, prime p)

π_* of a $K(n)$-local commutative E-algebra is a **graded amplified Γ-ring**:

- Γ is a certain twisted bialgebra over E_0.
- $Q_0 \in \Gamma$ and θ such that $Q_0(x) = x^p + p\,\theta(x)$.
- $\pi_* L_{K(n)} \mathbb{P}_E(\Sigma^q E) \approx F^\wedge_{\mathfrak{m}}$,
  $F =$ free graded amplified Γ-ring on one generator in dim. q.

Questions / topics

1. How does the formal group of E produce Γ? (Ando, Hopkins, Strickland)
2. What is the algebraic structure of Γ? (quadratic? Koszul?) (R.)

E = even periodic ring spectrum $\Longrightarrow$ formal group G_E.

Formal group G_E of E

Formal scheme $G_E = \mathrm{Spf}(E^0\mathbb{CP}^\infty)$ over $\pi_0 E$.
Group law $G_E \times G_E \to G_E$ defined by

$$\mu^*\colon E^0\mathbb{CP}^\infty \to E^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \approx E^0\mathbb{CP}^\infty \widehat{\otimes}_{E_0} E^0\mathbb{CP}^\infty.$$

$\mu\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifies $\otimes$ of line bundles.

Additive and multiplicative transformation of functors:

$$E^0(X) \xrightarrow{\psi} F^0(X)$$

ψ^* = homomorphism of formal groups over F_0,
where $g = \psi\colon E^0(*) \to F^0(*)$.

Part II: Formal groups and power operations (E-theory)

G_0/k formal group of height n, $E = E_{G_0/k}$. Power map:

$$E^0X \xrightarrow{P^m} E^0(X \times B\Sigma_m)$$

$$E^0X \xrightarrow{P^m} E^0(X) \otimes_{E_0} E^0(B\Sigma_m)$$

$$E^0X \xrightarrow{P^m} E^0(X) \otimes_{E_0} E^0(B\Sigma_m) \xrightarrow{\tau} E^0X \otimes_{E_0} E^0B\Sigma_m/I$$

Künneth isomorphism: $E^0B\Sigma_m$ is finite and flat over E_0.
I is the "transfer ideal":

$$I = \sum_{0<i<m} \mathrm{Image}\left[E^0B(\Sigma_i \times \Sigma_{m-i}) \xrightarrow{\text{transfer}} E^0B\Sigma_m\right].$$

Proposition

$\tau P^m\colon E^0X \to E^0X \otimes_{E^0} E^0B\Sigma_m/I$ is a ring homomorphism.

Remark: $E^0B\Sigma_m/I = 0$ unless $m = p^r$.

Part II: The associated homomorphism

Let $(F_{p^r})^0(X) = E^0 X \otimes_{E^0} E^0 B\Sigma_{p^r}/I$.

Ring homomorphisms:

- $s^*\colon E_0 \to (F_{p^r})_0$, induced by $B\Sigma_{p^r} \to *$.
- $t^*\colon E_0 \to (F_{p^r})_0$, defined by $\tau P^{p^r}\colon E^0(*) \to E^0(*) \otimes_{E_0} E^0(B\Sigma_{p^r})/I$.

The ring operation

$$E^0(X) \xrightarrow{\tau P^{p^r}} (F_{p^r})^0(X)$$

produces a homomorphism of formal groups defined over $(F_{p^r})_0$.

What kind of homomorphism?

Part II: Deformations of Frobenius

Frobenius. $\phi\colon k \to k$ defined by $\phi(x) = x^p$.
Relative Frobenius. $\mathrm{Frob}\colon G_0 \to \phi^* G_0$.

Definition

A **deformation of Frobenius** $(G, i, \psi) \to (G', i', \psi')$ (of deformations of G_0 to R) is a homomorphism $f\colon G \to G'$ of formal groups over R, such that

$$\begin{array}{ccc} \pi^* G & \xrightarrow{\ \pi^*(f)\ } & \pi^* G' \\ \psi \big\downarrow \wr & & \wr \big\downarrow \psi' \\ i^* G_0 & \xrightarrow[\ i^*(\mathrm{Frob}^r)\ ]{} & i'^* G_0 \end{array} \qquad \begin{array}{ccccc} & & R/\mathfrak{m} & & \\ & {}^{i}\nearrow & & \nwarrow^{i'} & \\ k & & \xleftarrow[\ \phi^r\ ]{} & & k \end{array}$$

commute for some $r \geq 0$.
($\pi\colon R \to R/\mathfrak{m}$.)

Remark: Deformations of Frobenius with domain (G, i, ψ) correspond *exactly* to finite subgroup schemes of G. ($f \rightsquigarrow \mathrm{Ker}(f) \subset G$.)

Part II: Descent (Ando-Hopkins-Strickland (mid 90s?))

$$\mathcal{D}(R) = \begin{cases} \text{Objects: deformations } (G, i, \phi) \text{ of } G_0/k \text{ to } R, \\ \text{Morphisms: deformations of Frobenius.} \end{cases}$$

$$f\colon R \to R' \quad \Longrightarrow \quad f^*\colon \mathcal{D}(R) \to \mathcal{D}(R').$$

Definition

A **sheaf of modules** M on $\mathcal{D} = \{\mathcal{D}(R)\}$ consists of

- functors $M_R\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathrm{Mod}_R$,
- natural isomorphisms $M_f\colon R' \otimes_R M_R \xrightarrow{\sim} M_{R'} \circ f^*$,

satisfying obvious "coherence" axioms.
$\Longrightarrow$ symmetric monoidal category $\mathrm{Mod}_{\mathcal{D}}$ of sheaves of modules.

Let Γ = ring of **additive power operations** for E.
That is, $\Gamma \subset \bigoplus_{m\geq 0} E_0^{\wedge} B\Sigma_m$ consisting of α such that Q_α is additive.

Theorem

Equivalence $\mathrm{Mod}_{\mathcal{D}} \approx \mathrm{Mod}_\Gamma$ *of symmetric monoidal categories.*

Part II: Strickland's Theorem

Operations $\tau P^{p^r}: E^0(X) \to (F_{p^r})^0(X) \implies$ homomorphism of formal groups $(\tau P^{p^r})^*: s^* G_E \to t^* G_E$ over $(F_{p^r})_0$.
Theorem depends on the following.

Claim

The homomorphism $(\tau P_{p^r})^*: s^* G_E \to t^* G_E$ over $(F_{p^r})_0$ is the universal example of a deformation of Frob^r between deformations of G_0.

$$(\text{Deformations } G \to G' \text{ of } \mathrm{Frob}^r) \iff (\text{subgroups } H \subset G \text{ of rank } p^r).$$

Result amounts to:

Theorem (Strickland (1998))

The data $(s^* G_E, \mathrm{Ker}(\tau P_{p^r})^*)$ over $(F_{p^r})_0 = E^0 B\Sigma_{p^r}/I$ is the universal example of a pair (G, H) consisting of a deformation G of G_0 and a finite subgroup scheme $H \subset G$ of rank p^r.

Recall: $\Gamma =$ ring of power operations for $E = E_{G_0/k}$.

- $\Gamma \subset \bigoplus_m E_0^\wedge B\Sigma_m$ is the submodule of primitives.
- $\Gamma = \bigoplus_r \Gamma_r$, where $\Gamma_r \subset E_0^\wedge B\Sigma_{p^r}$.
- Each Γ_r is a finitely generated free E_0-module, and

$$E^0 B\Sigma_{p^r}/I \approx \mathrm{Hom}_{E_0}(\Gamma_r, E_0).$$

- Each Γ_r is a cocommutative coalgebra $\Leftrightarrow$ product on $E^0 B\Sigma_{p^r}/I$.
- Associative product $\Gamma_r \otimes_{E_0} \Gamma_{r'} \to \Gamma_{r+r'} \Leftrightarrow$ composition of power operations.
- Warning: E_0 is not in the center of Γ.

Part III: Koszul algebras

$A = \bigoplus_{r \geq 0} A_r$ graded associative ring, $A_0 = R$ commutative.

Definition

A is **Koszul** if there exist R-modules C_r with $C_0 = R$, and an exact sequence (a "Koszul complex")

$$\cdots \xrightarrow{d} A \otimes_R C_3 \xrightarrow{d} A \otimes_R C_2 \xrightarrow{d} A \otimes_R C_1 \xrightarrow{d} A \otimes_R C_0 \xrightarrow{d} R \to 0$$

of left A-modules such that d raises degree by 1.

Fact

If A is Koszul, then

$$A \approx T_R(A_1)/(U), \qquad U \subset A_2$$

(i.e., A is "quadratic".)

Part III: Koszul algebras (Example 2)

- Back to the example: $\Gamma \approx \bigoplus \Gamma_r \approx T_{E_0}(\Gamma_1)/(U)$, where $\Gamma_1 = E_0\{Q_0, Q_1, Q_2\}$, $U =$ Adem relations.
  Note: Γ_1 is an E_0-bimodule; right E_0-module structure is determined by formulas $Q_i a = \cdots$ given earlier.
- **PBW Theorem** (Priddy (1970)): if Γ has a "nice" admissible basis, then Γ is Koszul.
- $\implies$ Exact sequence.

$$0 \to \Gamma \otimes_{E_0} C_2 \to \Gamma \otimes_{E_0} C_1 \to \Gamma \to E_0 \to 0.$$

C_i are free modules over E_0: $\operatorname{rank} C_1 = 3$, $\operatorname{rank} C_2 = 2$.

Part III: Is Γ always Koszul?

Conjecture (Ando-Hopkins-Strickland (mid 90s?))

For all $E = E_{G_0/k}$, the associated ring Γ of power operations is Koszul. The associated Koszul complex has the form

$$0 \to \Gamma \otimes_{E_0} C_n \to \cdots \to \Gamma \otimes_{E_0} C_1 \to \Gamma \to E_0 \to 0,$$

where $n =$ height of G_0.

- They developed a program to prove the result, using interesting ideas about a kind of "Bruhat-Tits building" formed using flags of certain finite subgroup schemes of G_E.
- I don't believe they ever completed their program; there may be no obstruction to doing so, however.
- There is another proof, which avoids using formal group theory; it uses ideas related to the Whitehead conjecture (Kuhn, Mitchell, Priddy) and calculus (Arone-Mahowald, Arone-Dwyer).

Here are some of the ideas in the proof.

Definition

Given a (nonadditive) functor $F\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$, the **linearization** $\mathcal{L}[F]\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$ is

$$\mathcal{L}[F](M) = \mathrm{Cok}\left[\; F(M \oplus M) \underset{F(\pi_1)+F(\pi_2)}{\overset{F(\pi_1+\pi_2)}{\rightrightarrows}} F(M) \;\right].$$

$\mathcal{L}[F]$ is initial additive quotient functor of F.

In some cases, including ours, $\mathcal{L}[F \circ G] \to \mathcal{L}[F] \circ \mathcal{L}[G]$ is an isomorphism.

The free amplified Γ-ring monad

- $F\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$ the free amplified Γ-ring functor.
- $M =$ an E-module; $F(\pi_0 M)$ "approximates"

$$\pi_0 L_{K(n)} \mathbb{P}_E(M) \approx \pi_0 L_{K(n)} \left(\bigvee_m (\underbrace{M \wedge_E \cdots \wedge_E M}_{m \text{ copies}})_{h\Sigma_m} \right).$$

- More precisely: for E-module M with $\pi_* M =$ flat E_*-module concentrated in even degree,

$$F(\pi_0 M) \approx \bigoplus_{m \geq 0} \pi_0 L_{K(n)} \mathbb{P}_E^m(M).$$

- Similarly, $(F \circ \cdots \circ F)(\pi_0 M)$ "approximates" $\pi_0(\mathbb{P} \circ \cdots \circ \mathbb{P})(M)$.

Apply linearization to $F \circ \cdots \circ F$.

- $$\mathcal{L}[F](E_0) = \Delta.$$

  $\Delta \approx \bigoplus_r \Delta_r$, where
  $\Delta_r \approx \mathrm{Cok}\left[\bigoplus_{0<i<p^r} E_0^\wedge B(\Sigma_i \times \Sigma_{p^r-i}) \to E_0^\wedge B\Sigma_{p^r}\right].$
  Δ is a ring, non-canonically isomorphic to Γ. (We actually show Δ is Koszul.)

- $$\mathcal{L}[F \circ \cdots\cdots \circ F](E_0) = \Delta \otimes_{E_0} \cdots \otimes_{E_0} \Delta.$$

- Monadic bar construction $\mathcal{B}_\bullet(F, F, F)$.

$$\mathcal{L}\left[\mathcal{B}_\bullet(F, F, F)\right] \approx \mathcal{B}_\bullet(\Delta, \Delta, \Delta).$$

(Priddy 1970):

- If Δ is a graded ring, filter $\mathcal{B}_\bullet(M, \Delta, N)$ according to grading on Δ.
- Δ is **Koszul** if $\mathrm{gr}_q \mathcal{B}_\bullet(E_0, \Delta, E_0)$ has homology concentrated in degree q.
- Koszul complex "is" the spectral sequence associated to this filtration on $\mathcal{B}_\bullet(M, \Delta, N)$; $E_1^{p,q} =$ chain complex.

$$\mathcal{B}_q(F,F,F)(E_0) \approx (\underbrace{F \circ \cdots \circ F}_{(q+2)\text{ times}})(E_0) \approx \bigoplus_{m \geq 0} E_0^{\wedge}(K_q(m)_{h\Sigma_m}).$$

$K_\bullet(m)$ is the **partition complex**.

- A partition of $\underline{m} = \{1, \ldots, m\}$ is an equivalence relation E on $\underline{m}$.
- Partitions ordered by refinement: $E \leq E' \Leftrightarrow E$ finer than E'.
- $$K_\bullet(m) = \mathrm{nerve}\,\{\text{poset of partitions of } \underline{m}\}\,.$$
- $$K_q(m) = \{(E_0 \leq E_1 \leq \cdots \leq E_q)\}.$$

Apply linearization to partition description of $\mathcal{B}_\bullet(F,F,F)$ to get

$$\mathcal{B}_q(\Delta,\Delta,\Delta) \approx \mathcal{L}[\mathcal{B}_q(F,F,F)](E_0) \approx \bigoplus_{m\geq 0} Q_m(K_q(m))$$

where

$$Q_m(X) = \mathrm{Cok}\left[\bigoplus_{0<i<m} E_0^\wedge(X_{h(\Sigma_i\times\Sigma_{m-i})}) \to E_0^\wedge(X_{h\Sigma_m})\right],$$

X is a set with Σ_m action.
Facts about Q_m:

- $Q_m(X \amalg Y) \approx Q_m(X) \oplus Q_m(Y)$.
- $Q_m(\Sigma_m/H) = 0$ if H does *not* act transitively on $\underline{m}$.

- Let $\overline{K}_\bullet(m) = K_\bullet(m)/K^\diamond_\bullet(m)$, where

$$K^\diamond_q(m) = \{ (E_0 \leq \cdots \leq E_q) \mid E_0 \text{ not finest } \textit{or } E_q \text{ not coarsest} \}.$$

- Then

$$\mathcal{B}_\bullet(E_0, \Delta, E_0) \approx \bigoplus_m Q_m(\overline{K}_\bullet(m)).$$

- This is 0 unless $m = p^r$, in which case we want to show:

$$\mathrm{gr}_{p^r}\mathcal{B}_\bullet(E_0, \Delta, E_0) \approx Q_{p^r}(\overline{K}_\bullet(p^r)) \text{ has } H_* \text{ concentrated in degree } r.$$

Part III: The idea of the proof

- Need to show $Q_{p^r}(\overline{K}_\bullet(p^r))$ has H_* concentrated in degree r.
- $$K_\bullet(p^r) \times \Sigma_{p^r}/(\Sigma_p \wr \cdots \wr \Sigma_p) \twoheadrightarrow K_\bullet(p^r),$$

  where

  $$U_\bullet(p^r) = \bigcup_{\substack{A \subset \Sigma_{p^r} \\ \text{max. ab. subgp.}}} (K_\bullet(p^r) \times \Sigma_{p^r}/(\Sigma_p \wr \cdots \wr \Sigma_p))^A .$$
- Can form analogous quotient $\overline{U}_\bullet(p^r)$.
- Reduce to showing $Q_{p^r}(\overline{U}_\bullet(p^r))$ is chain homotopy equivalent to a complex concentrated in degree r.
- Claim: There is a Σ_{p^r}-equivariant homotopy equivalence $\overline{U}_\bullet(p^r) \approx X_+ \wedge S^r$, where X is some Σ_{p^r}-set.

Part III: $U_\bullet(p^r)$ and the Tits building for $GL(r, \mathbb{F}_p)$

- $A \subset \Sigma_{p^r}$ maximal abelian subgroup:

$$K_\bullet(p^r)^A = \mathrm{nerve}\,\{\text{ poset of subgroups of } A\ \}\,.$$

For $A \approx (\mathbb{Z}/p)^r$, the quotient $\overline{K}_\bullet(p^r)^A$ is (a 2-fold suspension of) the Tits building for $GL(r, \mathbb{F}_p)$.

- $$\overline{K}_\bullet(p^r)^A \approx \begin{cases} \bigvee S^r & \text{if } A \approx (\mathbb{Z}/p)^r, \\ * & \text{otherwise.} \end{cases}$$

$A = (\mathbb{Z}/p)^r$ result is theorem of Solomon-Tits (1969).
Non-elementary A: can be shown in exactly the same way.

- Show $\overline{U}_\bullet(p^r) \approx X_+ \wedge S^r$ (Σ_{p^r}-equivariantly) by the same "shellability" argument that Solomon-Tits use for $\overline{K}_\bullet(p^r)^{(\mathbb{Z}/p)^r}$.

- The "shellability" argument gives an explicit chain homotopy $H\colon \mathrm{id} \sim f$ of maps of normalized chain complexes $NQ_{p^r}(\overline{U}_\bullet(p^r)) \to NQ_{p^r}(\overline{U}_\bullet(p^r))$, where

$$f = 0 \text{ when } \bullet \neq r.$$

- HKR $\Longrightarrow Q_{p^r}(\overline{K}_q(p^r) \wedge \Sigma_{p^r}/\Sigma_p^{\wr r}{}_+) \approx Q_{p^r}(\overline{U}_q(p^r)) \oplus (p\text{-torsion})$.
- Get chain homotopy $H'\colon \mathrm{id} \sim f'$ on $NQ_{p^r}(\overline{K}_\bullet(p^r) \wedge \Sigma_{p^r}/\Sigma_p^{\wr r}{}_+))$ by "extending by 0", so that

$$p^{\text{large}} f' = 0 \text{ when } \bullet \neq r.$$

- $NQ_{p^r}(\overline{K}_\bullet(p^r))$ is retract of $NQ_{p^r}(\overline{K}_\bullet(p^r) \wedge \Sigma_{p^r}/\Sigma_p^{\wr r}{}_+)$ (transfer), and is p-torsion free.
- Get desired chain homotopy $H''\colon \mathrm{id} \sim f''$ on $NQ_{p^r}(\overline{K}_\bullet(p^r))$, with $f'' = 0$ if $\bullet \neq r$.

The end

http://www.math.uiuc.edu/~rezk/baltimore-2010-power-ops-handout.pdf

- Ando, "Isogenies of formal group laws and powen operations in the cohomology theories E_n", Duke 1995.
- Ando, Hopkins, Strickland, "The sigma orientation is an H_∞ map", Amer. J. 2004; arXiv:math/0204053.
- Bruner, May, McClure, Steinberger, *H_∞ ring spectra and their applications*, LNM 1176, 1986.
- Hovey, Strickland, *Morava K-theories and localization*, Mem. AMS 139, 1999.
- Kashiwabara, "$K(2)$-homology of some infinite loop spaces", Math. Z., 1995.
- Priddy, "Koszul resolutions", Trans. AMS, 1970.
- Rezk, "The units of a ring spectrum and a logarithmic cohomology operation", J. AMS, 2006; arXiv:math/0407022.
- Rezk, "Power operations for Morava E-theory of height 2 at the prime 2", arXiv:0812.1320.
- Rezk, "The congruence criterion for power operations in Morava E-theory", arXiv:0902.2499.
- Strickland, "Finite subgroups of formal groups", J. Pure Appl. Alg., 1997.
- Strickland, "Morava E-theory of symmetric groups", Topology, 1998; arXiv:math/9801125.
- Strickland, "$K(n)$-local duality for finite groups and groupoids", Topology, 2000; arXiv:math/0011109.