FREE COLIMIT COMPLETION IN ∞-CATEGORIES

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Abstract. We show how several useful properties of Ind-constructions in ∞-categories extend to arbitrary free colimit completion constructions.

1. Introduction

It is well-known that the ∞-category $\mathrm{PSh}(C)$ of presheaves of ∞-groupoids on C is the "free colimit completion" of C. More generally, there is a "free $\mathcal{F}$-colimit completion" $\mathrm{PSh}^{\mathcal{F}}(C)$ for a given class $\mathcal{F}$ of ∞-categories, which can be exhibited as the full subcategory of $\mathrm{PSh}(C)$ generated by representable presheaves under $\mathcal{F}$-colimits, as described by Lurie in [Lur09, 5.3.6]. Note that "free" here means we are not merely adjoining some colimits, but rather that the construction is characterized by a universal property.

In special cases we have more. For instance, when $\mathcal{F}$ is the class of *κ-filtered* ∞-categories for some regular cardinal κ, then the free $\mathcal{F}$-colimit completion admits a rather more explicit description: it is $\mathrm{Ind}_\kappa(C)$, the full subcategory of presheaves X on C which represent a right fibration $C/X \to C$ such that C/X is κ-filtered. Here C/X is the "point category" of the functor $X\colon C^{\mathrm{op}} \to \mathcal{S}$.

Furthermore, there is a very useful "recognition principle" for such categories: any ∞-category A which is generated under κ-filtered colimits by a full subcategory $C \subseteq A$ of "κ-compact objects" is canonically equivalent to $\mathrm{Ind}_\kappa(C)$ [Lur09, 5.3.5]. These Ind-categories are the basis of the theory of accessible ∞-categories.

This note addresses the question: to what extent can these pleasant properties of $\mathrm{Ind}_\kappa(C)$ be extended to arbitrary free colimit completions? The answer is: in some sense, pretty much all of them. Our results are encapsulated by the following observation.

Whether a presheaf X is in $\mathrm{PSh}^{\mathcal{F}}(C)$ depends only on its point category C/X.

Here is a brief summary.

- Any class $\mathcal{F}$ of small ∞-categories can be enlarged to a *filtering closure* $\overline{\mathcal{F}}$, which consists of C whose free $\mathcal{F}$-colimit completion $\mathrm{PSh}^{\mathcal{F}}(C)$ contains a terminal presheaf (§5). We say that $\mathcal{F}$ is a *filtering class* when $\mathcal{F} = \overline{\mathcal{F}}$ (§4).
- We get an explicit criterion for describing the free $\mathcal{F}$-colimit completion $\mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ as a full subcategory, much as for $\mathrm{Ind}_\kappa(C)$: a presheaf X is in $\mathrm{PSh}^{\mathcal{F}}(C)$ if and only if C/X is in the filtering closure of $\mathcal{F}$, where $C/X \to C$ is the right fibration classified by X (5.2).
- Thus free $\mathcal{F}$-colimit completion depends only on the filtering closure $\overline{\mathcal{F}}$, and in fact the filtering classes precisely correspond to possible "types" of free colimit completion (6.7).
- We find that any ∞-category which has all $\mathcal{F}$-colimits also has all $\overline{\mathcal{F}}$-colimits, and any functor between such which preserves all $\mathcal{F}$-colimits also preserves all $\overline{\mathcal{F}}$-colimits (6.4), (6.5).
- In the context of fully faithful functors, the previous is in some sense the best possible: $\overline{\mathcal{F}}$ is the largest class for which we can have such a result (6.2).
- The ∞-groupoids in a filtering class $\overline{\mathcal{F}}$, as well as the the underlying weak homotopy types of objects of $\overline{\mathcal{F}}$, are precisely those in the full subcategory of ∞-groupoids generated by the terminal object under $\mathcal{F}$-colimits (7.1).

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- There is a recognition principle for free $\mathcal{F}$-colimit completion generalizing that for Ind-categories, which is stated in terms of the evident notion of *$\mathcal{F}$-compact object*. (9.2).
- It is straightforward to produce examples of filtering classes, which include the familiar classes of *κ-filtered* and *sifted* ∞-categories, but also many others which have not been much studied (§8, §10).

Given all this, it would be very desirable to have methods for calculating filtering closures of various classes of interest. Further study is needed!

I came to this while working on a project to understand generalizations of Ind-constructions and accessible ∞-categories, motivated by the 1-categorical work of [ABLR02]. The idea is to look at classes $\mathcal{F}$ of ∞-categories characterized by how $\mathcal{F}$-colimits of ∞-groupoids preserve a fixed collection of types of limit [Rez21], e.g., much as κ-filtered colimits preserve κ-small limits of ∞-groupoids, or sifted colimits preserve finite products of ∞-groupoids. In the course of this I realized that for many purposes there is nothing special about classes described in terms of such limit preservation. It is fair to say that none of the results here are particularly deep, but the picture they make is pleasant and perhaps surprising. I note that there is every reason to expect that most of the ∞-categorical results described here have 1-categorical analogues. However, I have not attempted to trace this out explicitly.

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2. Basic ∞-categorical notions

2.1. **Universes.** We work with respect to a chosen **universe** of small simplicial sets, which determines an ∞-category Cat_∞ of **small ∞-categories**, together with a full subcategory $\mathcal{S} \subseteq \mathrm{Cat}_\infty$ of **small ∞-groupoids**. We say that an ∞-category is **locally small** if its mapping spaces are equivalent to small ∞-groupoids.

We also discuss ∞-categories which are not small. These may be imagined to live in some higher universe, but I will not need to refer explicitly to a hierarchy of universes as in [Lur09, 1.2.15]. However, some of the results I use do rely on universe-hopping, most notably Lurie's construction of free colimit completions (3.3), and his related embedding theorem (3.4).

2.2. **Colimits.** By a **small colimit**, I mean a colimit of a functor $J \to A$ where J is a small ∞-category. I say that an ∞-category A is **cocomplete** if it has all *small* colimits, and **complete** if it has all *small* limits.

More generally given a class $\mathcal{F}$ of ∞-categories I will speak of **$\mathcal{F}$-colimits**, i.e., colimits of functors $J \to A$ where $A \in \mathcal{F}$. Thus, I can speak of an ∞-category A which **has $\mathcal{F}$-colimits**, i.e., is such that every $J \to A$ with $J \in \mathcal{F}$ admits a colimit.

Given a fully faithful functor $f\colon A \to B$ (e.g., the inclusion of a full subcategory), I say that f is **stable under $\mathcal{F}$-colimits** if (i) both A and B have $\mathcal{F}$-colimits and (ii) f preserves all $\mathcal{F}$-colimits.

2.3. **Presheaves.** Given a small ∞-category C, we write $\mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ for the category of **presheaves** of ∞-groupoids on C (rather than Lurie's notation $\mathcal{P}(C)$ of [Lur09, 5.1]). I denote the Yoneda functor by $\rho_C\colon C \to \mathrm{PSh}(C)$, or just ρ if the context is clear. Recall that ρ is fully faithful and that $\mathrm{PSh}(C)$ is complete and cocomplete.

2.4. **Slices of presheaves.** Given a presheaf $X \in \mathrm{PSh}(C)$, I write

$$C/X := C \times_{\mathrm{PSh}(C)} \mathrm{PSh}(C)_{/X}$$

for the evident pullback of the slice projection $\mathrm{PSh}(C)_{/X} \to \mathrm{PSh}(C)$ along ρ, and $\pi_X\colon C/X \to C$ for the evident projection. The composite $\rho_C\pi_X\colon C/X \to \mathrm{PSh}(C)$ comes with an extension to a colimit functor $\widetilde{\rho}\colon (C/X)^{\triangleright} \to \mathrm{PSh}(C)_{/X}$, which exhibits X *tautologically* as a colimit of $\rho\pi_X$ [Lur09, 5.1.5.3].

In particular, the colimit of $\rho\colon C \to \mathrm{PSh}(C)$ is a terminal presheaf, since $C/1 \approx C$.

2.5. *Remark.* C/X may be regarded as an "∞-category of elements" or **point category** of X, by analogy with the 1-categorical analogue. The projection $\pi_X\colon C/X \to C$ is a right fibration, representing the unstraightening of the functor $X\colon C^{\mathrm{op}} \to \mathcal{S}$.

The evident functor $C/X \to \mathrm{PSh}(C)_{/X}$ induces by restriction an equivalence

$$\kappa\colon \mathrm{PSh}(C)_{/X} \to \mathrm{PSh}(C/X),$$

i.e., every slice of a presheaf category is a presheaf category on a category of elements [Lur09, 5.1.6.12].

Finally, note that if $X \in \mathrm{PSh}(C)$ and $\widetilde{Y} := (f\colon Y \to X) \in \mathrm{PSh}(C)_{/X}$, then we have an equivalence

$$(\mathrm{PSh}(C)_{/X})_{/\widetilde{Y}} \approx \mathrm{PSh}(C)_{/Y},$$

which when combined with the equivalence $\kappa\colon \mathrm{PSh}(C)_{/X} \approx \mathrm{PSh}(C)$ restricts to an equivalence of full subcategories

$$(C/X)/\widetilde{Y} \approx C/Y.$$

3. Free colimit completion

The Yoneda functor $\rho\colon C \to \mathrm{PSh}(C)$ exhibits the **free colimit completion** of C.

3.1. **Theorem.** [Lur09, 5.1.5.6] *For any cocomplete ∞-category A, restriction along ρ induces an equivalence*

$$\mathrm{Fun}(\mathrm{PSh}(C), A) \supseteq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C), A) \to \mathrm{Fun}(C, A)$$

from the category of colimit preserving functors $\mathrm{PSh}(C) \to A$ to the category of functors $C \to A$.

In particular, any functor $f\colon C \to A$ admits an essentially unique extension $\widehat{f}\colon \mathrm{PSh}(C) \to A$ to a colimit preserving functor equipped with a natural isomorphism $\widehat{f}\rho \approx f$.

As a consequence, $\mathrm{PSh}(C)$ contains the **universal C-colimit**, which is just the terminal presheaf.

3.2. **Corollary.** *Let $f\colon C \to A$ be any functor from a small ∞-category to a cocomplete ∞-category. Then the colimit of f in A is equivalent to $\widehat{f}(1)$, where $\widehat{f}\colon \mathrm{PSh}(C) \to A$ is any colimit preserving extension of f along ρ.*

Proof. Since $\widehat{f}\colon \mathrm{PSh}(C) \to A$ preserves colimits, and $1 \approx \mathrm{colim}_C\, \rho$. □

Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Given a small ∞-category C, let $\mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ denote the full subcategory **generated by representable presheaves under $\mathcal{F}$-colimits**. That is, $\mathrm{PSh}^{\mathcal{F}}(C)$ is the smallest full subcategory of presheaves which (i) contains the image of the Yoneda functor $\rho\colon C \to \mathrm{PSh}(C)$ and (ii) is stable under $\mathcal{F}$-colimits. The restriction $\rho\colon C \to \mathrm{PSh}^{\mathcal{F}}(C)$ exhibits the **free $\mathcal{F}$-colimit completion** of C.

3.3. **Theorem.** [Lur09, 5.3.6.2] *If $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ is a class of small ∞-categories, and if A is an ∞-category which has $\mathcal{F}$-colimits, then restriction along ρ exhibits an equivalence*

$$\mathrm{Fun}(\mathrm{PSh}^{\mathcal{F}}(C), A) \supseteq \mathrm{Fun}^{\mathcal{F}-\mathrm{colim}}(\mathrm{PSh}^{\mathcal{F}}(C), A) \to \mathrm{Fun}(C, A)$$

from the category of $\mathcal{F}$-colimit preserving functors $\mathrm{PSh}^{\mathcal{F}}(C) \to A$ to $\mathrm{Fun}(C, A)$.

In particular, any functor $f\colon C \to A$ admits an essentially unique extension $\widehat{f}\colon \mathrm{PSh}^{\mathcal{F}}(C) \to A$ to an $\mathcal{F}$-colimit preserving functor equipped with a natural isomorphism $\widehat{f}\rho \approx f$.

We refer to Lurie for the proof, but note that his proof both provides and relies on the following, which we will use later.

3.4. **Theorem** (Embedding theorem)**.** [Lur09, 5.3.6.2] *Given any classes $\mathcal{F} \subseteq \mathcal{G}$ of ∞-categories and an ∞-category A which has $\mathcal{F}$-colimits, there exists a fully faithful functor $i\colon A \rightarrowtail B$ such that (i) B has all $\mathcal{G}$-colimits and (ii) i preserves all $\mathcal{F}$-colimits.*

Sketch proof. Construct B as a full subcategory (in fact, a localization) of $\mathrm{Fun}(A^{\mathrm{op}}, \widehat{\mathcal{S}})$ where $\mathcal{S}$ is an ∞-category of ∞-groupoids in a suitably large universe. □

4. Filtering classes

Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Note that if $C \in \mathcal{F}$ then necessarily $\mathrm{PSh}^{\mathcal{F}}(C)$ contains the terminal presheaf, since the terminal presheaf is the colimit of ρ_C (2.4).

We say that $\mathcal{F}$ is **filtering** if the converse is also true, i.e., if for every small ∞-category C, we have that $C \in \mathcal{F}$ whenever $\mathrm{PSh}^{\mathcal{F}}(C)$ contains the terminal presheaf.

4.1. *Remark.* Since $\mathrm{PSh}^{\mathcal{F}}(C)$ contains every representable presheaf, any terminal object of it is necessarily equivalent to the terminal presheaf of $\mathrm{PSh}(C)$. So in the definition of filtering class we could equally say "$C \in \mathcal{F}$ whenever $\mathrm{PSh}^{\mathcal{F}}(C)$ contains a terminal object".

For a filtering classes we have an explicit criterion for identifying objects of $\mathrm{PSh}^{\mathcal{F}}(C)$.

4.2. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. The following are equivalent.*

(1) *For all $C \in \mathrm{Cat}_\infty$ and $X \in \mathrm{PSh}(C)$, we have that $X \in \mathrm{PSh}^{\mathcal{F}}(C)$ if and only if $C/X \in \mathcal{F}$.*
(2) *$\mathcal{F}$ is a filtering class.*

Proof. (1) $\implies$ (2). If $1 \in \mathrm{PSh}(C)$ is a terminal presheaf then $C/1 \approx C$. Specializing (1) to such objects says that $1 \in \mathrm{PSh}^{\mathcal{F}}(C)$ implies $C \in \mathcal{F}$, which is precisely the condition that $\mathcal{F}$ is filtering.

(2) $\implies$ (1). For any $C \in \mathrm{Cat}_\infty$, define the full subcategory

$$\mathrm{Ind}^{\mathcal{F}}(C) := \{\, X \in \mathrm{PSh}(C) \mid C/X \in \mathcal{F} \,\}.$$

Note that we always have $\mathrm{Ind}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}^{\mathcal{F}}(C)$, even if $\mathcal{F}$ is not filtering, since every presheaf $X \in \mathrm{PSh}(C)$ is tautologically a colimit of the composite $C/X \xrightarrow{\pi} C \xrightarrow{\rho} \mathrm{PSh}(C)$ (2.4).

I will show that if $\mathcal{F}$ is filtering, then (a) $\rho(C) \subseteq \mathrm{Ind}^{\mathcal{F}}(C)$, and (b) $\mathrm{Ind}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ is stable under $\mathcal{F}$-colimits. This implies that $\mathrm{Ind}^{\mathcal{F}}(C) = \mathrm{PSh}^{\mathcal{F}}(C)$, which is precisely condition (1).

(a) If C is any small ∞-category with a terminal object t, then $\rho(t) \in \mathrm{PSh}(C)$ is a terminal presheaf contained in $\mathrm{PSh}^{\mathcal{F}}(C)$. Since $\mathcal{F}$ is filtering it contains all such C. The claim follows since for any representable presheaf $\rho(c) \in \mathrm{PSh}(C)$ the category $C/\rho(c) \approx C_{/c}$ has a terminal object.

(b) Let $f' \colon J^{\triangleright} \to \mathrm{PSh}(C)$ be a colimit diagram such that $J \in \mathcal{F}$ and $f = f'|J$ takes values in $\mathrm{Ind}^{\mathcal{F}}(C)$, and write $X := f'(v) \in \mathrm{PSh}(C)$ for the value at the cone point, i.e., the colimit of f. To show that $X \in \mathrm{Ind}^{\mathcal{F}}(C)$ we must show that $C/X \in \mathcal{F}$, i.e., that $\mathrm{PSh}^{\mathcal{F}}(C/X)$ contains the terminal presheaf.

Under join/slice-adjunction the functor f' corresponds to a functor $g' \colon J \to \mathrm{PSh}(C)_{/X}$, whose colimit is the terminal object of the target. Let $g \colon J \to \mathrm{PSh}(C/X)$ be the composite of g' with the equivalence $\kappa \colon \mathrm{PSh}(C)_{/X} \xrightarrow{\sim} \mathrm{PSh}(C/X)$ (2.4). I will show that $g(J) \subseteq \mathrm{Ind}^{\mathcal{F}}(C/X)$, and therefore that $g(J) \subseteq \mathrm{PSh}^{\mathcal{F}}(C/X)$ since $\mathrm{Ind}^{\mathcal{F}}(C/X) \subseteq \mathrm{PSh}^{\mathcal{F}}(C/X)$. The claim will follow since the colimit of g in $\mathrm{PSh}(C/X)$ is the terminal object.

To finish the proof, we use the fact that if $f \colon Y \to X$ is an object of $\mathrm{PSh}(C)_{/X}$, and $\widetilde{Y} := \kappa(Y) \in \mathrm{PSh}(C/X)$ is its image, then we have an equivalence (2.4)

$$(C/X)/\widetilde{Y} \approx C/Y.$$

Thus $Y \in \mathrm{Ind}^{\mathcal{F}}(C)$ if and only if $\widetilde{Y} \in \mathrm{Ind}^{\mathcal{F}}(C/X)$. Therefore $f(J) \subseteq \mathrm{Ind}^{\mathcal{F}}(C)$ implies $g(J) \subseteq \mathrm{Ind}^{\mathcal{F}}(C/X)$ as desired. □

5. Filtering closure

Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be any class of small ∞-categories. The **filtering closure** of $\mathcal{F}$ is defined to be the class

$$\overline{\mathcal{F}} := \{\, C \in \mathrm{Cat}_\infty \mid \mathrm{PSh}^{\mathcal{F}}(C) \text{ contains a terminal presheaf} \,\}.$$

5.1. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. We have the following.*

(1) *$\mathcal{F} \subseteq \overline{\mathcal{F}}$.*
(2) *For any small ∞-category C, the $\mathcal{F}$-colimit completion $\mathrm{PSh}^{\mathcal{F}}(C)$ of C is stable under $\overline{\mathcal{F}}$-colimits of presheaves.*
(3) *For any small ∞-category C, we have $\mathrm{PSh}^{\mathcal{F}}(C) = \mathrm{PSh}^{\overline{\mathcal{F}}}(C)$.*

Proof. (1) If $C \in \mathcal{F}$, then $1 \approx \mathrm{colim}_C\, \rho \in \mathrm{PSh}^{\mathcal{F}}(C)$, whence $C \in \overline{\mathcal{F}}$.

(2) Suppose $f\colon J \to \mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ is some functor where $J \in \overline{\mathcal{F}}$. By the universal property of free $\mathcal{F}$-colimit completion (3.3) f extends along $\rho\colon J \rightarrowtail \mathrm{PSh}^{\mathcal{F}}(J)$ to an $\mathcal{F}$-colimit preserving functor $\widehat{f}\colon \mathrm{PSh}^{\mathcal{F}}(J) \to \mathrm{PSh}^{\mathcal{F}}(C)$. Since $J \in \overline{\mathcal{F}}$ we have $1 \in \mathrm{PSh}^{\mathcal{F}}(J)$, whence $\widehat{f}(1) \in \mathrm{PSh}^{\mathcal{F}}(C)$. We are done, since $\widehat{f}(1)$ is exactly the colimit of f in $\mathrm{PSh}(C)$ (3.2).

(3) Since $\mathcal{F} \subseteq \overline{\mathcal{F}}$ by (1) we have $\mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}^{\overline{\mathcal{F}}}(C)$, and thus (2) implies equality. □

Using this we get a general criterion for identifying objects of $\mathrm{PSh}^{\mathcal{F}}(C)$.

5.2. **Corollary.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Then $X \in \mathrm{PSh}^{\mathcal{F}}(C)$ if and only if $1 \in \mathrm{PSh}^{\mathcal{F}}(C/X)$.*

Proof. Immediate from (5.1)(3), the criterion (4.2)(1), and the definition of $\overline{\mathcal{F}}$. □

5.3. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Then $\overline{\mathcal{F}}$ is a filtering class, and is in fact the smallest filtering class containing $\mathcal{F}$. In particular, $\mathcal{F}$ is filtering if and only if $\mathcal{F} = \overline{\mathcal{F}}$.*

Proof. We have already noted (5.1)(1) that $\mathcal{F} \subseteq \overline{\mathcal{F}}$. That $\overline{\mathcal{F}}$ is a filtering class is immediate from (5.1)(3), since $C \in \overline{\mathcal{F}}$ if and only if $1 \in \mathrm{PSh}^{\mathcal{F}}(C) = \mathrm{PSh}^{\overline{\mathcal{F}}}(C)$.

Suppose $\mathcal{F}'$ is any filtering class which contains $\mathcal{F}$. Then for any $J \in \overline{\mathcal{F}}$ we have

$$1 \in \mathrm{PSh}^{\overline{\mathcal{F}}}(C) = \mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}^{\mathcal{F}'}(C),$$

and thus $\overline{\mathcal{F}} \subseteq \mathcal{F}'$.

The final claim is immediate. □

In the following I write $\mathrm{PSh}^{\mathcal{F}}(C)_{/X} := \mathrm{PSh}^{\mathcal{F}}(C) \times_{\mathrm{PSh}(C)} \mathrm{PSh}(C)_{/X}$. This is a slight abuse of notation, since X might not be an object of $\mathrm{PSh}^{\mathcal{F}}(C)$.

5.4. **Corollary.** *For any class $\mathcal{F} \in \mathrm{Cat}_\infty$, any $C \in \mathrm{Cat}_\infty$, and any presheaf $X \in \mathrm{PSh}(C)$, the equivalence $\kappa\colon \mathrm{PSh}(C)_{/X} \to \mathrm{PSh}(C/X)$ identifies the full subcategories $\mathrm{PSh}^{\mathcal{F}}(C)_{/X}$ and $\mathrm{PSh}^{\mathcal{F}}(C/X)$.*

Proof. We can use the criterion (5.2) to determine whether objects are contained in $\mathrm{PSh}(C)^{\mathcal{F}}$ and $\mathrm{PSh}(C/X)^{\mathcal{F}}$. The claim follows immediately using the observation (2.4) that if κ sends $f\colon Y \to X$ to $\widetilde{Y}$, then $C/Y \approx (C/X)/\widetilde{Y}$. □

5.5. *Remark.* We note that there is a more direct proof of (5.4), which in turn gives another route to proving (5.2), since the conclusion of (5.4) immediately implies that $X \in \mathrm{PSh}^{\mathcal{F}}(C)$ if and only if $1 \in \mathrm{PSh}^{\mathcal{F}}(C/X)$.

Here is a sketch of a direct proof of (5.4). Given a fully faithful functor $r\colon C \rightarrowtail A$ with A cocomplete, let $A_r^{\mathcal{F}} \subseteq A$ be the smallest subcategory containing $r(C)$ and stable under $\mathcal{F}$-colimits. This is a union

$$A_r^{\mathcal{F}} = \bigcup_\lambda A_r^\lambda$$

of subcategories indexed in an evident way by ordinals, i.e., so that $A^0 = r(C)$, $A^\lambda = \bigcup_{\mu<\lambda} A^\mu$ if λ is a limit ordinal, and $A_r^{\lambda+1}$ is the full subcategory of A spanned by the objects of A_r^λ together with all objects which are colimits in A of functors $J \to A_r^\lambda$ with $J \in \mathcal{F}$, Now suppose given a pullback square of ∞-categories

$$\begin{array}{ccc} C & \overset{r}{\rightarrowtail} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ C' & \underset{r'}{\rightarrowtail} & A' \end{array}$$

such that r and r' are fully faithful and A' is cocomplete, and with with the property that a functor $f\colon J^{\triangleright} \to A$ is a colimit if and only if uf is a colimit. Then it is straightforward to show by induction on λ that $A^{\mathcal{F}}_{r} = u^{-1}(A'^{\mathcal{F}}_{r'})$. Then apply this to $u\colon \mathrm{PSh}(C)_{/X} \to \mathrm{PSh}(C)$ and $p\colon C/X \to C$.

6. Properties of the filtering closure

6.1. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and let $A' \subseteq A$ be a full subcategory of a cocomplete ∞-category A. If A' is stable under $\mathcal{F}$-colimits, then it is also stable under $\overline{\mathcal{F}}$-colimits.*

Proof. Given any functor $f\colon J \to A'$ with $J \in \overline{\mathcal{F}}$, we want to show that the colimit of f in A is actually in A'. By the universal property f extends over $\rho\colon J \rightarrowtail \mathrm{PSh}(J)$ to a colimit preserving functor $\widehat{f}\colon \mathrm{PSh}(J) \to A$, so that the colimit of f in A is equivalent to $\widehat{f}(1)$ (3.2). Furthermore, $\widehat{f}(\mathrm{PSh}^{\mathcal{F}}(J)) \subseteq A'$ since A' is stable under $\mathcal{F}$-colimits. The claim follows since $J \in \overline{\mathcal{F}}$ so $1 \in \mathrm{PSh}^{\mathcal{F}}(J)$. □

In a certain sense the above result is the best possible.

6.2. **Corollary.** *Suppose $\mathcal{F}, \mathcal{G} \subseteq \mathrm{Cat}_\infty$ are classes of small ∞-categories, with the property that for any full subcategory $A' \subseteq A$ of a cocomplete ∞-category A, whenever A' is stable under $\mathcal{F}$-colimits then it is also stable under $\mathcal{G}$-colimits. Then $\mathcal{G} \subseteq \overline{\mathcal{F}}$.*

Proof. Suppose $J \in \mathcal{G}$ and consider $A' := \mathrm{PSh}^{\mathcal{F}}(J)$ and $A := \mathrm{PSh}(J)$. By hypothesis A' is stable under $\mathcal{G}$-colimits in A, and thus in particular $1 \approx \mathrm{colim}_J\, \rho \in \mathrm{PSh}^{\mathcal{F}}(J)$, so $J \in \overline{\mathcal{F}}$ as desired. □

We can expand on the statement of (6.1), using Lurie's embedding theorem.

6.3. **Corollary.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and let $A' \subseteq A$ be a full subcategory where A has all $\overline{\mathcal{F}}$-colimits. If A' is stable under $\mathcal{F}$-colimits, then it is also stable under $\overline{\mathcal{F}}$-colimits.*

Proof. Choose an $\overline{\mathcal{F}}$-colimit preserving embedding $A \rightarrowtail B$ to a cocomplete ∞-category (3.4) and apply (6.1) to $A' \rightarrowtail B$. □

6.4. **Corollary.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Then any ∞-category A which has $\mathcal{F}$-colimits also has $\overline{\mathcal{F}}$-colimits.*

Proof. Choose an $\overline{\mathcal{F}}$-colimit preserving embedding $A \rightarrowtail B$ to a cocomplete ∞-category (3.4) and apply (6.1). □

6.5. **Corollary.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and let $f\colon A' \to A$ be a functor between categories which have all $\mathcal{F}$-colimits. If f preserves $\mathcal{F}$-colimits then f preserves $\overline{\mathcal{F}}$ colimits.*

Proof. This is immediate from (6.3) and (11.2) (proved later), which says that f preserves $\mathcal{G}$-colimits if and only if the full subcategory $\mathrm{Path}(f) \subseteq \mathrm{LPath}(f)$ is stable under $\mathcal{G}$-colimits. □

The same ideas give another characterization of filtering closure.

6.6. **Lemma.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and let $J \in \mathrm{Cat}_\infty$. Then $J \in \overline{\mathcal{F}}$ if and only if $\mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ is stable under J-colimits for all $C \in \mathrm{Cat}_\infty$.*

Proof. We have already shown that $J \in \overline{\mathcal{F}}$ implies stability (5.1). For the converse, take $C = J$ and recall that $1 \approx \mathrm{colim}_J\, \rho_J$. □

6.7. **Corollary.** *Given $\mathcal{F}, \mathcal{G} \in \mathrm{Cat}_\infty$, we have that $\overline{\mathcal{F}} = \overline{\mathcal{G}}$ if and only if $\mathrm{PSh}^{\mathcal{F}}(C) = \mathrm{PSh}^{\mathcal{G}}(C)$ for all $C \in \mathrm{Cat}_\infty$.*

Finally, we note some "closure properties" of filtering closure.

6.8. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories.*

1. *If $u\colon J \to K$ is a cofinal functor between small ∞-categories, then $J \in \mathcal{F}$ implies $K \in \overline{\mathcal{F}}$.*
2. *If $J, K \in \mathcal{F}$ then $J \times K \in \overline{\mathcal{F}}$.*

Proof. For (1), note that if u is cofinal then $u^*\colon \mathrm{Fun}(K^{\triangleright}, A) \to \mathrm{Fun}(J^{\triangleright}, A)$ (restriction along u) preserves all colimit cones [Lur09, 4.1.1.8]. In particular if $\mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ is stable under J-colimits, then it is also stable under K-colimits, so the claim follows from (6.6).

For (2), note that $J \times K$-colimits can be computed as the composite

$$\mathrm{Fun}(J \times K, \mathrm{PSh}(C)) = \mathrm{Fun}(J, \mathrm{Fun}(K, \mathrm{PSh}(C))) \xrightarrow{\mathrm{colim}_J} \mathrm{Fun}(K, \mathrm{PSh}(C)) \xrightarrow{\mathrm{colim}_K} \mathrm{PSh}(C).$$

Since colimits in functor categories are computed objectwise, if $J, K \in \mathcal{F}$ then $\mathrm{PSh}^{\mathcal{F}}(C)$ is stable under $J \times K$-colimits, whence the claim follows from (6.6). □

6.9. *Remark.* It is *not* the case that filtering closures are "generated by cofinality". Simple counterexamples include $\mathcal{F} = \varnothing$ (10.1) and $\mathcal{F} = \{\Delta^0 \amalg \Delta^0\}$ (10.4).

7. Filtering classes and ∞-groupoids

Given an ∞-category C, we write $\eta\colon C \to |C|$ for a tautological map to its groupoid completion. Given a class $\mathcal{F} \subseteq \mathrm{Cat}_\infty$, we write $\mathcal{S}^{\mathcal{F}} := \mathrm{PSh}(1)^{\mathcal{F}}$.

7.1. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Then*

$$\overline{\mathcal{F}} \cap \mathcal{S} = \left|\overline{\mathcal{F}}\right| = \mathcal{S}^{\mathcal{F}},$$

i.e., the class of ∞-groupoids in $\overline{\mathcal{F}}$ is the class of groupoid completions of objects of $\overline{\mathcal{F}}$, which are the objects of the full subcategory of ∞-groupoids generated under $\mathcal{F}$-colimits by the terminal object.

Proof. Clearly $\overline{\mathcal{F}} \cap \mathcal{S} \subseteq \left|\overline{\mathcal{F}}\right|$. The groupoid completion map $\eta\colon J \to |J|$ is cofinal (7.2), so $J \in \overline{\mathcal{F}}$ implies $|J| \in \overline{\mathcal{F}}$ (6.8)(1), whence $\left|\overline{\mathcal{F}}\right| \subseteq \overline{\mathcal{F}} \cap \mathcal{S}$. We know that $X \in \mathrm{PSh}(1)^{\mathcal{F}}$ if and only if $X \approx X/1 \in \overline{\mathcal{F}}$ (4.2), so $\mathcal{S}^{\mathcal{F}} = \overline{\mathcal{F}} \cap \mathcal{S}$ □

7.2. **Lemma.** *The tautological map $\eta\colon C \to |C|$ from an ∞-category to its group completion is cofinal.*

Proof. It suffices to prove this for a particular model of η. For instance, there exists a factorization $C \xrightarrow{j} C' \xrightarrow{p} \Delta^0$ into a right anodyne map j followed by a right fibration p. Since the target of p is the terminal object, it is actually a Kan fibration, so C' is an ∞-groupoid, while j is a weak equivalence of simplicial sets. Thus j is a groupoid completion of C, and it is a cofinal map since all right anodyne maps are cofinal [Lur09, 4.1.1.3]. □

8. Constructing filtering classes

8.1. **Proposition.** *Let $\{A_i' \subseteq A_i\}$ be a collection of pairs of ∞-categories, where each A_i' is a full subcategory of a cocomplete ∞-category A_i, and define*

$$\mathcal{F} := \{\, C \in \mathrm{Cat}_\infty \mid A_i' \text{ is stable under } C\text{-colimits in } A_i \text{ for all } i \,\}.$$

Then $\mathcal{F}$ is a filtering class.

Proof. We need to show that $1 \in \mathrm{PSh}^{\mathcal{F}}(C)$ implies $C \in \mathcal{F}$, so suppose $1 \in \mathrm{PSh}^{\mathcal{F}}(C)$. Given any functor $f\colon C \to A'_i \subseteq A_i$ for some i, consider the $\mathcal{F}$-colimit preserving extension $\widehat{f}\colon \mathrm{PSh}^{\mathcal{F}}(C) \to A_i$ of f. By hypothesis the image of $\widehat{f}$ is contained in A'_i, and since $1 \in \mathrm{PSh}^{\mathcal{F}}(C)$ we have $\widehat{f}(1) \in A'_i$. Since this object is isomorphic to the colimit of f in A_i, we have shown any such colimit is in A'_i. Thus every A'_i is stable under C-colimits, so $C \in \mathcal{F}$. □

In the situation of the previous proposition, we will say that the filtering class $\mathcal{F}$ is **cut out** by the collection of embeddings $\{A'_i \subseteq A_i\}$. It is easy to see that every filtering class $\mathcal{F}$ arises in this way: it is cut out by $\{\mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)\}_{C \in \mathrm{Cat}_\infty}$, since if $\mathrm{PSh}^{\mathcal{F}}(J) \subseteq \mathrm{PSh}(J)$ is stable under J-colimits then $1 \in \mathrm{PSh}^{\mathcal{F}}(J)$.

8.2. **Corollary.** *The intersection of any collection of filtering classes is a filtering class.*

9. A recognition principle for free colimit completion

Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and A an ∞-category which has $\mathcal{F}$-colimits. Say that an object a of A is **$\mathcal{F}$-compact** if

$$\mathrm{Map}_A(a, -)\colon A \to \mathcal{S}$$

preserves all $\mathcal{F}$-colimits. I write $A^{\mathcal{F}-\mathrm{cpt}} \subseteq A$ for the full subcategory of $\mathcal{F}$-compact objects.

The notion of $\mathcal{F}$-compactness really only depends on the filtering closure of $\mathcal{F}$.

9.1. **Proposition.** *Let A be an ∞-category which has $\mathcal{F}$-colimits. Then an object of A is $\mathcal{F}$-compact if and only if it is $\overline{\mathcal{F}}$-compact.*

Proof. Apply (6.4) and (6.5) to A and to $\mathrm{Map}_A(a, -)$. □

Thus, if a is $\mathcal{F}$-compact, then $\mathrm{Map}_A(a, -)$ preserves $\overline{\mathcal{F}}$-colimits.

9.2. **Proposition.** *Let $\mathcal{F} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories, and suppose $C \in \mathrm{Cat}_\infty$. Let $\widehat{f}\colon \mathrm{PSh}^{\mathcal{F}}(C) \to A$ be an $\mathcal{F}$-colimit preserving functor to an ∞-category which has $\mathcal{F}$-colimits, and let $f = \widehat{f}\rho_C\colon C \to A$.*

1. *If f is fully-faithful and $f(C) \subseteq A^{\mathcal{F}-\mathrm{cpt}}$, then $\widehat{f}$ is fully faithful.*
2. *The functor $\widehat{f}$ is an equivalence if and only if*
   - (i) *f is fully faithful.*
   - (ii) *$f(C) \subseteq A^{\mathcal{F}-\mathrm{cpt}}$.*
   - (iii) *The objects of $f(C)$ generate A under $\mathcal{F}$-colimits.*

Proof. Without loss of generality we can replace $\mathcal{F}$ with $\overline{\mathcal{F}}$, using (6.4), (5.1), (6.5), and (9.1). Furthermore, we know that $\mathrm{PSh}^{\mathcal{F}}(C) = \mathrm{Ind}^{\mathcal{F}}(C) = \{\, X \in \mathrm{PSh}(C) \mid C/X \in \overline{\mathcal{F}} \,\}$ (5.2), and that every X is tautologically a C/X-colimit of representable presheaves.

Then this is proved exactly as in [Lur09, 5.3.5.11], which deals with the special case where $\mathcal{F}$ is the class of κ-filtered ∞-categories for some regular cardinal κ. □

9.3. *Remark.* Note that the original example $C \xrightarrow{\rho} \mathrm{PSh}^{\mathcal{F}}(C) \subseteq \mathrm{PSh}(C)$ of a free $\mathcal{F}$-colimit completion is exactly of this type, since all representable presheaves are "completely compact" [Lur09, 5.1.6.2]. In the case of $\mathcal{F} = \mathrm{Cat}_\infty$ this recovers [Lur09, 5.1.6.11].

10. Examples of filtering classes and filtering closures

10.1. **The minimal filtering class.** Since $\mathrm{PSh}^{\varnothing}(C) \approx C$, we have that

$$\overline{\varnothing} = \{\, C \in \mathrm{Cat}_\infty \mid C \text{ has a terminal object} \,\}.$$

10.2. **The maximal filtering class.** Clearly Cat_∞ is a filtering class, and $\mathrm{PSh}^{\mathrm{Cat}_\infty}(C) = \mathrm{PSh}(C)$. The Cat_∞-compact objects of a cocomplete ∞-category are precisely what are called *completely compact* in [Lur09, 5.1.6.2].

10.3. **Coproducts.** Let $\mathrm{Set} \subseteq \mathrm{Cat}_\infty$ be the collection of all small and discrete ∞-groupoids, so that $\mathrm{PSh}^{\mathrm{Set}}(C)$ is the free completion of C with respect to small coproducts. It is straightforward to show that $\mathrm{PSh}^{\mathrm{Set}}(C)$ consists exactly of presheaves which are equivalent to small coproducts of representables, as this subcategory is itself clearly stable under coproducts. Thus $1 \in \mathrm{PSh}^{\mathrm{Set}}(C)$ implies $1 \approx \coprod_i \rho(c)$, and using this you can show that

$$\overline{\mathrm{Set}} = \{\, \coprod_i C_i \mid \text{each } C_i \in \mathrm{Cat}_\infty \text{ has a terminal object} \,\}.$$

Similar considerations identify $\overline{\mathrm{Set}^{<\kappa}}$, the filtering closure of κ-small sets, where κ is any regular cardinal.

10.4. **Binary coproducts.** Let $\mathcal{F} = \{\Delta^0 \amalg \Delta^0\}$, so that $\mathrm{PSh}^{\mathcal{F}}(C)$ is the free completion of C with respect to pairwise coproducts. Then

$$\overline{\mathcal{F}} = \{\, \coprod_{i\in I} C_{/c_i} \mid I \text{ is finite and non-empty, and each } C_i \in \mathrm{Cat}_\infty \text{ has a terminal object} \,\}.$$

10.5. **Generalized filtered categories.** Let $\mathcal{U} \subseteq \mathrm{Cat}_\infty$ be a class of small ∞-categories. Say that $J \in \mathrm{Cat}_\infty$ is **$\mathcal{U}$-filtering** if $\mathrm{colim}_J \colon \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves all $\mathcal{U}$-limits, where $\mathcal{S}$ denotes the ∞-category of ∞-groupoids. Write $\mathrm{Filt}_{\mathcal{U}} \subseteq \mathrm{Cat}_\infty$ for the class of all $\mathcal{U}$-filtering ∞-categories. Then $\mathrm{Filt}_{\mathcal{U}}$ is a filtering class, since it is cut out (8.1) by the collection of inclusions

$$\mathrm{Fun}^{\text{limit cones}}(U^{\triangleleft}, \mathcal{S}) \subseteq \mathrm{Fun}(U^{\triangleleft}, \mathcal{S}),$$

where U ranges over all categories in $\mathcal{U}$. Such classes are studied in [Rez21]. What follows are a few special cases of this.

10.6. **Idempotent completion.** Let Idem be the walking idempotent. Then $\mathrm{PSh}^{\{\mathrm{Idem}\}}(C)$ is an idempotent completion of C [Lur09, 5.3.6.9]. Thus

$$\overline{\{\mathrm{Idem}\}} = \{\, C \in \mathrm{Cat}_\infty \mid \text{the idempotent completion of } C \text{ has a terminal object} \,\}.$$

It can be shown that $\overline{\{\mathrm{Idem}\}} = \mathrm{Filt}_{\mathrm{Cat}_\infty}$, the class of $J \in \mathrm{Cat}_\infty$ such that J-colimits of ∞-groupoids preserve all small limits.

10.7. **κ-filtered ∞-categories.** Given a regular cardinal κ, let Sm_κ denote the class of κ-small ∞-categories, i.e., ones which are equivalent to a κ-small simplicial set. Consider the filtering class $\mathrm{Filt}_{\mathrm{Sm}_\kappa}$, which consists of all $J \in \mathrm{Cat}_\infty$ such that J-colimits of ∞-groupoids preserve finite limits.

Then $\mathrm{Filt}_{\mathrm{Sm}_\kappa}$ is precisely the collection of all small *κ-filtered* ∞-categories [Lur09, 5.3.1.7], i.e., those J such that $K \to J$ extends over $K \subseteq K^{\triangleright}$ for every κ-small simplicial set K. (See [Lur09, 5.3.3.3] for a proof.)

In particular, (4.2) says that $\mathrm{PSh}^{\mathrm{Filt}_{\mathrm{Sm}_\kappa}}(C) = \mathrm{Ind}_\kappa(C)$, where the latter is as in [Lur09, 5.3.5]. Objects are $\mathrm{Filt}_{\mathrm{Sm}_\kappa}$ compact if and only if they are κ-compact in the usual sense [Lur09, 5.3.4]. Thus, our recognition principle (9.2) for free κ-filtered-colimit completion recovers Lurie's [Lur09, 5.3.5.11].

10.8. **Sifted ∞-categories.** Let $\mathrm{Set}^{<\omega} \subseteq \mathrm{Cat}_\infty$ denote the class of finite discrete ∞-groupoids. Consider the filtering class $\mathrm{Filt}_{\mathrm{Set}^{<\omega}}$, which consists of all $J \in \mathrm{Cat}_\infty$ such that J-colimits of ∞-groupoids preserve finite products.

Then $\mathrm{Filt}_{\mathrm{Set}^{<\omega}}$ is precisely the collection of all small *sifted* ∞-categories [Lur09, 5.5.8.1], i.e., those J such that (i) J is non-empty, and (ii) the diagonal $\delta\colon J \to J \times J$ is cofinal. (This equivalence is well-known. The main part of the proof is [Lur09, 5.5.8.11-12].)

Sifted colimit completion of C is studied in [Lur09, 5.5.8], in the special case when C itself is assumed to have finite coproducts.

10.9. **Distilled ∞-categories.** Consider the filtering class $\mathrm{Filt}_{\{\Lambda^2_2\}}$, which consists of all $J \in \mathrm{Cat}_\infty$ such that J-colimits of ∞-groupoids preserve pullbacks.

Then $\mathrm{Filt}_{\{\Lambda^2_2\}}$ is exactly the class of small *distilled* ∞-categories, where we say that J is **distilled** if for every functor $f\colon \Lambda^2_0 \to J$, the slice $J_{f/}$ has contractible weak homotopy type. This identification is proved in [Rez21]. Results of that paper show that the class of small distilled ∞-categories is the filtering closure of $\mathrm{Filt}_{\mathrm{Sm}_\omega} \cup \mathcal{S}$.

10.10. **Weakly contractible ∞-categories.** The filtering class $\mathrm{Filt}_{\{\varnothing\}}$ consists of J such that J-colimits of ∞-groupoids preserve the terminal object, i.e., such that $\mathrm{colim}_J *$ is contractible. These are precisely the weakly contractible ∞-categories.

10.11. **Other examples?** The problem of determining $\overline{\mathcal{F}}$ when $\mathcal{F}$ is not already known to be a filtering class is unexplored. Here are some interesting possibilities to consider.

- $\mathcal{F} = \mathcal{S}$, the class of ∞-groupoids. I offer a conjecture: $\overline{\mathcal{S}}$ should consist of C such that the map $u\colon C \to |C|$ to its groupoid completion is a left adjoint (or what is the same thing: such that there exists a cofinal functor $G \to C$ from an ∞-groupoid).
- $\mathcal{F} = \mathrm{Sm}_\omega$, the class of ω-small ∞-categories, so that $C \to \mathrm{PSh}^{\mathrm{Sm}_\omega}(C)$ is a free finite-colimit completion of C. I have no sense here of what $\overline{\mathrm{Sm}_\omega}$ should look like.

11. Functors preserving colimits

In this section I prove a criterion for preservation of colimits by functors which is surely well-known, but for which I have no convenient reference. This proof was suggested to me by Maxime Ramzi.

Given a functor $f\colon A \to B$ of ∞-categories, the **lax path category** of f is

$$\mathrm{LPath}(f) := (B \times A) \times_{B\times B} \mathrm{Fun}(\Delta^1, B),$$

so that objects of $\mathrm{LPath}(f)$ correspond to triples $(b, a, \gamma\colon b \to f(a))$ with a an object of A, b an object of B, and γ a morphism of B. We write $\mathrm{Path}(f) \subseteq \mathrm{LPath}(f)$ for the **path category**, i.e., the full subcategory spanned by (b, a, γ) such that γ is an isomorphism in B. We write $\pi_A\colon \mathrm{LPath}(f) \to A$ and $\pi_B\colon \mathrm{LPath}(f) \to B$ for the evident projection functors.

11.1. **Lemma.** *Let K be a simplicial set, and let $g\colon K \to \mathrm{LPath}(f)$ be a map. If $\pi_A g$ and $\pi_B g$ admit colimits in A and B respectively, then g has a colimit, and both π_A and π_B preserve such colimits.*

11.2. **Corollary.** *Let J be an ∞-category, and suppose $f\colon A \to B$ is a functor between ∞-categories which have J-colimits. The following are equivalent.*

1. *The functor f preserves all J-colimits.*
2. *The full subcategory $\mathrm{Path}(f) \subseteq \mathrm{LPath}(f)$ is stable under J-colimits.*

Proof of Corollary from the Lemma. (1) $\Longrightarrow$ (2). Suppose $g\colon J^{\triangleright} \to \mathrm{LPath}(f)$ is a colimit diagram such that $g(J) \subseteq \mathrm{Path}(f)$. Using (11.1) we see that g corresponds to a triple (β, α, γ) where $\alpha\colon J^{\triangleright} \to A$ and $\beta\colon J^{\triangleright} \to B$ are colimit diagrams, and $\gamma\colon \beta \to f\alpha$ is a natural transformation of functors $J^{\triangleright} \to B$ such that $\gamma|_J$ is a natural isomorphism. Since by hypothesis f preserves J-colimits, $f\alpha\colon J^{\triangleright} \to B$ is also colimit diagram, so γ must be an isomorphism, i.e., $g(J^{\triangleright}) \subseteq \mathrm{Path}(f)$.

(2) $\Longrightarrow$ (1). Suppose $\alpha\colon J^{\triangleright} \to A$ is a colimit diagram. Since B has J-colimits, we can construct a map $g\colon J^{\triangleright} \to \mathrm{LPath}(f)$ corresponding to the triple (β, α, γ), where $\beta\colon J^{\triangleright} \to B$ is a colimit diagram, and $\gamma\colon \beta \to f\alpha$ is such that $\gamma|_J$ is an isomorphism of functors $J \to B$. By (11.1) the map g is a colimit diagram such that $g(J) \subseteq \mathrm{Path}(f)$, and thus by hypothesis $g(J^{\triangleright}) \subseteq \mathrm{Path}(f)$, whence γ is an isomorphism of functors, so $f\alpha$ is a colimit as desired. □

Sketch proof of the Lemma. We can use the same idea as the proof of [NS18, II.1.5(iv)]. If (b, a, γ) and (b', a', γ') are objects of $\mathrm{LPath}(f)$, then the space of maps of the first to the second is isomorphic to the pullback of the diagram

$$\mathrm{Map}_{\mathrm{Fun}(\Delta^1, B)}(\gamma, \gamma') \to \mathrm{Map}_B(b, b') \times \mathrm{Map}_B(fa, fa') \xleftarrow{\mathrm{id} \times f} \mathrm{Map}_B(b, b') \times \mathrm{Map}_A(a, a').$$

of simplicial sets. This is seen to be equivalent to the pullback of a diagram the form

$$\mathrm{Map}_B(b, b') \xrightarrow{\gamma'_*} \mathrm{Map}_B(b, fa') \xleftarrow{\gamma^* \circ f} \mathrm{Map}_A(a, a').$$

Now use the mapping space criterion for colimits [Lur09, 4.2.4.3] to show that a map $h\colon K^{\triangleright} \to \mathrm{LPath}(f)$ is a colimit if both $\pi_A h$ and $\pi_B h$ are colimits.

It remains to show that any $g\colon K \to \mathrm{LPath}(f)$ admits a colimit h of the above form if $\pi_A g$ and $\pi_B g$ admit colimits. This is straightforward: choose colimits $h_A\colon K^{\triangleright} \to A$ of $\pi_A g$ and $h_B\colon K^{\triangleright} \to B$ of $\pi_B g$. The universal property of colimits provides a natural transformation $\eta\colon h_B \to f h_A$ of functors $K^{\triangleright} \to B$ extending the given transformation $\gamma\colon \pi_B g \to f\pi_A g$, and thus we can take h to be the map corresponding to the triple (h_B, h_A, η). □

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