DEGENERATE EDGES OF CARTESIAN FIBRATIONS ARE CARTESIAN EDGES

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Abstract. We prove the following fact which is difficult to find in the literature: if $p\colon X \to S$ is a Cartesian fibration of arbitrary simplicial sets, and f is a degenerate edge of X, then f is a p-Cartesian edge.

1. Introduction

The purpose of this note is to establish the following fact (5.4): degenerate edges of Cartesian fibrations are Cartesian edges. (This of course implies the "opposite" fact: degenerate edges of coCartesian fibrations are coCartesian edges.)

That degenerate edges of Cartesian fibrations are Cartesian edges is seemingly required for much of the material in Chapters 2–4 of [Lur09] (for instance [3.1.1.8]), but apparently not explicitly proved there. Furthermore, the situation is somewhat delicate: as pointed out by Alexander Campbell (see (2.1) below), degenerate edges of inner fibrations need not be Cartesian or coCartesian edges, contrary to what one might have assumed.

The proof given here originally arose from a discussion in the Homotopy Theory chatroom on `mathoverflow.net`, on or around September 14–15, 2016, involving myself, Denis Nardin, and Dylan Wilson. I was later able to simplify that proof somewhat: it is not difficult, and mainly amounts to a combination of [2.1.4.5], [2.4.1.12], and [2.4.2.8]. The simplified proof is given in §5 and actually shows a little more: edges in Cartesian fibrations which are *isomorphisms in a fiber quasicategory* are always Cartesian edges (5.3). A version of the original proof is given in §6.

I have also given an exposition of some of the needed results from [Lur09], especially when doing so would seem to clarify things. For instance, the proofs rely on the implication (1) $\Longrightarrow$ (3) of [2.4.2.8] in [Lur09], but the proof of that given there is hard for me to follow. So I have given a proof here (5.2). Furthermore, the original proof of (5.4) relied on implication (3) $\Longrightarrow$ (1) of [2.4.2.7], whose proof there is quite terse. I have give a more detailed proof of that implication (6.1) here, modulo a piece of that argument which is broken out as a separate proposition (7.1), which is of independent interest.

I had completely forgotten about the 2016 discussion and proof until recently, when Alexander Campbell raised the question again in the same chatroom. This led me to search for earlier chatroom discussions related to this, by which I rediscovered the 2016 discussion. I'm writing this up and putting it online, so that the next time I forget about the existence of this proof a Google search will be likely to find it for me.

Thanks to the participants of the aforementioned chatroom discussion, as well as Alexander Campbell and David Breugmann, who commented on versions of this document.

All purely numerical citations (e.g., "[2.4.1.1]") are to numbered statements in [Lur09].

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2. Cartesian edges

Let $p\colon X \to S$ be a map of simplicial sets. An edge $f\colon x \to y$ of X is **p-Cartesian** [2.4.1.1] if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a trivial fibration.

I'm going to refer to these kinds of edges generically as "Cartesian edges" when I don't want to refer to a particular map p. But remember that the definition is meaningful only in the context of a given map p.

The practice of [Lur09], whenever speaking of p-Cartesian edges, is to require in addition that p be an inner fibration. I won't insist on this, as the definition makes sense for general maps p, and some elementary facts about p-Cartesian edges do not rely on p being an inner fibration. However, in practice there is not much to say about p-Cartesian edges unless p has some additional property, which typically includes being an inner fibration.

Here is an equivalent formulation: an edge f is p-Cartesian if and only if for every $n \geq 2$ and every commutative square

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{u} & X \\ \downarrow & \overset{s}{\nearrow} & \downarrow p \\ \Delta^n & \xrightarrow[v]{} & S \end{array}$$

such that $u(\langle n-1, n\rangle) = f$, a lift s exists [2.4.1.4].

2.1. *Remark.* It is *not* the case that a degenerate edge of an inner fibration $p\colon X \to S$ must be p-Cartesian. We give an example, which is the op-dual of the example of [Cam19].

Let $S = \Delta^2/(\langle 12\rangle \sim *)$ be the quotient of the 2-simplex obtained by collapsing the edge $\langle 12\rangle$ to a point. Let $p\colon X \rightarrowtail S$ be the inclusion of the subcomplex generated by the image of the edge $\langle 02\rangle$, so that $X \approx \Delta^1$.

The map p is seen to be an inner fibration, as follows. There is a pullback square of the form

$$\begin{array}{ccc} \Lambda^2_2 & \xrightarrow{u} & X \\ q\downarrow & & \downarrow p \\ \Delta^2 & \xrightarrow[v]{} & S \end{array}$$

where v is the quotient map. The horn inclusion q is a map between (nerves of) 1-categories and thus an inner fibration. It is then straightforward to show that p is an inner fibration, using that v is surjective.

The edge $f = u(\langle 12\rangle)$ is degenerate in X, but no lift exists in the above square since p is not surjective, and thus f is not p-Cartesian.

As we will see, for an inner fibration $p\colon X \to S$ we have that: degenerate edges of X are p-Cartesian *if S is a quasicategory* (§3), and that degenerate edges of X are always *locally p-Cartesian* (4.3) (for any S). (For instance, the edge f of (2.1) is locally p-Cartesian though not p-Cartesian.)

Given a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ S' & \xrightarrow[v]{} & S \end{array}$$

I'll say that a simplex σ' of X' is a **pullback** of a simplex σ of X if $u(\sigma') = \sigma$, and that it is "the pullback over τ'" if $p'(\sigma) = \tau'$, where τ' is a simplex in S' such that $v(\tau') = p(\sigma)$.

It is immediate from the definitions that for such a pullback square, if f is a p-Cartesian edge, then any pullback f' of f is a p'-Cartesian edge [2.4.1.3 (2)]. That is, "Cartesian edges are preserved by base change".

The definition of Cartesian edge can thus be reformulated as follows.

2.2. **Proposition.** *If $p\colon X \to S$ is a map and f is an edge in X, then f is a p-Cartesian edge if and only if for every basechange $p'\colon X' \to \Delta^n$ of p along a map $v\colon \Delta^n \to S$ with $n \geq 2$ and $v(\langle n-1, n\rangle) = p(f)$, the pullback edge f' of f in X' over $\langle n-1, n\rangle$ is p'-Cartesian.*

When p is an inner fibration this can be sharpened significantly (7.2), as we will see.

2.3. *Warning.* It is *not* the case that a pullback f of a degenerate edge f' is also degenerate. (It is surprisingly easy to fall into the trap of assuming this.)

It *is* however true that the pullback of a degenerate edge *over a degenerate edge* is degenerate.

3. Cartesian edges in inner fibrations between quasicategories

We consider $p\colon X \to S$ an inner fibration to a quasicategory S. It follows that X is also a quasicategory. In this setting, isomorphisms are always Cartesian edges, by the "Joyal lifting theorem".

3.1. **Proposition** ([2.1.4.5], [Joy02])**.** *If $p\colon X \to S$ is an inner fibration between quasicategories, and f any edge in X, the following are equivalent: (i) f is an isomorphism in X. (ii) f is p-Cartesian and $p(f)$ is an isomorphism in S.*

Thus, if $p\colon X \to S$ is an inner fibration *between quasicategories*, then every isomorphism of X is p-Cartesian, and so in particular every degenerate edge (i.e., identity map) of X is p-Cartesian [2.1.4.6]. Furthermore, every p-Cartesian edge of X over a degenerate edge of S is an isomorphism in X.

Furthermore, in this context, "Cartesian edges are closed under composition and left cancellation".

3.2. **Proposition** ([2.4.1.7])**.** *Let $p\colon X \to S$ be an inner fibration between quasicategories, and consider a 2-simplex in X with edges*

$$\begin{array}{ccccc} & & y & & \\ & {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

If f is p-Cartesian, then g is p-Cartesian iff h is p-Cartesian

4. Locally Cartesian edges

Let $p\colon X \to S$ be a map of simplicial sets. An edge f of X is **locally p-Cartesian** [2.4.1.11] if the edge f' is p'-Cartesian, where $p'\colon X' \to \Delta^1$ is the base change of p along the map $\Delta^1 \to S$ representing the edge $p(f)$ in S, and f' is the pullback edge of f over $\langle 01\rangle$ in X'.

As in the Cartesian case, I'll speak generically of "locally Cartesian edges" when I don't want to specify a map p.

Here is an equivalent formulation: an edge f is locally p-Cartesian if and only if for every $n \geq 2$ and every commutative square

$$\begin{array}{ccccc} \Lambda^n_n & & \xrightarrow{\quad u \quad} & & X \\ \downarrow & & \overset{s}{\nearrow} & & \downarrow p \\ \Delta^n & \xrightarrow[\langle 0\cdots 01\rangle]{} & \Delta^1 & \xrightarrow[v]{} & S \end{array}$$

such that $u(\langle n-1, n\rangle) = f$, a lift s exists. This has the following immediate consequence.

4.1. **Proposition.** *Let $p\colon X \to \Delta^k$ be map of simplicial sets with $k \geq 1$, and let f be an edge of X such that $p(f) = \langle 0m\rangle$ for some $m \in [k]$. Then f is a p-Cartesian edge if and only if it is a locally p-Cartesian edge.*

Proof. The hypothesis implies that every commutative square we need to examine to prove f is p-Cartesian has as its bottom side the composite of maps

$$\Delta^n \xrightarrow{\langle 0\cdots 01\rangle} \Delta^1 \xrightarrow{\langle 0m\rangle} \Delta^k,$$

and f being locally p-Cartesian supplies us with lifts for such squares. □

Clearly, any p-Cartesian edge is locally p-Cartesian.

Here is another equivalent formulation: an edge f is locally p-Cartesian if there exists a basechange $p'\colon X' \to S'$ of p and a p'-Cartesian edge f' of X' which is a pullback of f. That this is the case just amounts to the fact that Cartesian edges are preserved by base change, and that any edge in X' such as f' is in the image of base change along some map $\Delta^1 \to S'$. In other words, "locally p-Cartesian" is in some sense a "local condition" with respect to the codomain of p.

It is clear that locally Cartesian edges are preserved under basechange, just as Cartesian edges are. In fact, if $p'\colon X' \to S'$ is a basechange of $p\colon X \to S$, and an edge f' of X' is a pullback of an edge f of X, then f' is locally p'-Cartesian *if and only if* f is locally p-Cartesian [2.4.1.12]. That is, "locally Cartesian edges are preserved and detected by basechange". This stronger detection property does not hold for Cartesian edges, since there are locally Cartesian edges which are not Cartesian (e.g., the edge f of (2.1)).

Given an inner fibration $p\colon X \to S$ we say that an edge f of X is a **fiber isomorphism** if (i) $p(f)$ is a degenerate edge on some vertex $v \in S$, and (ii) f is an isomorphism in the subcomplex $C = p^{-1}(v)$, which is a quasicategory since p is an inner fibration.

4.2. **Proposition.** *Let $p\colon X \to S$ be an inner fibration and f an edge of X which is a fiber isomorphism. Then f is locally p-Cartesian.*

Proof. We have a pullback square of the form

$$\begin{array}{ccc} C & \rightarrowtail & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^0 & \underset{v}{\rightarrowtail} & S \end{array}$$

with C a quasicategory since p and thus p' are inner fibrations. Since locally Cartesian edges are detected by basechange, to show f is locally p-Cartesian it suffices to show that its (unique) pullback f' in C is p'-Cartesian (and thus locally p'-Cartesian). This is in fact the case, since p' is an inner fibration between quasicategories and f' is an isomorphism in C (3.1). □

4.3. **Corollary.** *Let $p\colon X \to S$ be an inner fibration and f an degenerate edge in X. Then f is locally p-Cartesian.*

Proof. Immediate from (4.2), since any such degenerate edge is an identity map in its fiber quasicategory, and so a fiber isomorphism. □

The following asserts that locally Cartesian edges in an inner fibration which lie over the same edge of the base are in some sense equivalent.

4.4. **Proposition.** *Let $p\colon X \to S$ be an inner fibration, and let $h\colon x \to z$ and $f : y \to z$ be edges in X with the same target vertex, and such that $p(h) = p(f)$. If f and h are locally p-Cartesian, then there exists a 2-simplex in X of the form*

$$\begin{array}{ccccc} & & y & & \\ & {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

such that $p(g)$ is the degenerate edge on $v = p(x) = p(y)$ and g is an isomorphism in the fiber quasicategory $C = p^{-1}(v)$.

Proof. Let $p'\colon X' \to \Delta^1$ be the basechange of p along $\Delta^1 \to S$ representing the edge $p(f) = p(h)$, and let f' and h' be the evident pullback edges in X' over $\langle 01\rangle$, which therefore are p'-Cartesian edges. The induced map $X' \to X$ sends $p'^{-1}(\langle 0\rangle)$ isomorphically to $C = p^{-1}(v)$, so it suffices to construct a 2-simplex σ in X' with $\sigma\langle 12\rangle = f'$ and $\sigma\langle 02\rangle = h'$, so that $\sigma\langle 01\rangle = g'$ is an isomorphism in the fiber quasicategory $p'^{-1}(\langle 0\rangle)$.

In other words, to prove the claim we can assume without loss of generality, in addition to the hypotheses of the proposition, that $S = \Delta^1$, with $p(f) = p(h) = \langle 01\rangle$ and $v = \langle 0\rangle$, and that f and h are p-Cartesian edges.

In this situation, using that f is p-Cartesian we construct a 2-simplex σ in X lifting the 2-simplex $\langle 001\rangle$ in Δ^1 and such that $\sigma\langle 12\rangle = f$ and $\sigma\langle 02\rangle = h$. Since p is now an inner fibration between quasicategories and f and h are p-Cartesian, the edge $g = \sigma\langle 01\rangle$ is also p-Cartesian since in this case Cartesian edges are closed under left cancellation (3.2). Furthermore, since g is p-Cartesian and $p(g)$ is degenerate, we must have that g is an isomorphism in X (3.1).

Finally, note that the only simplices σ in $S = \Delta^1$ such that every vertex of σ is $\langle 0\rangle$ are the degenerate simplices on $\langle 0\rangle$. It follows that the data that witnesses that g is an isomorphism in X (i.e., the inverse edge, and homotopies of composites of g and its inverse with identity maps) all live in the fiber quasicategory $C = p^{-1}(\langle 0\rangle)$. Thus g is an isomorphism in C as desired. □

5. Cartesian fibrations

A map $p\colon X \to S$ is a **Cartesian fibration** [2.4.2.1] if

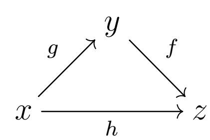
(1) p is an inner fibration, and
(2) for every edge $f\colon x \to y$ in S and vertex $\widetilde{y}$ in X such that $p(\widetilde{y}) = y$, there exists a p-Cartesian edge $\widetilde{f}\colon \widetilde{x} \to \widetilde{y}$ in X such that $p(\widetilde{f}) = f$.

It is clear that any basechange of a Cartesian fibration is again a Cartesian fibration [2.4.2.3 (2)].

If we have a Cartesian fibration *between quasicategories*, then there is no distinction between Cartesian and locally Cartesian edges.

5.1. **Proposition.** *Let $p\colon X \to S$ be a Cartesian fibration between quasicategories. Then any locally p-Cartesian edge in X is a p-Cartesian edge.*

Proof. Let $h\colon x \to z$ be a locally p-Cartesian edge. Since p is a Cartesian fibration, we can choose a p-Cartesian edge $f\colon y \to z$ with $p(h) = p(f)$, and note that f is thus also locally p-Cartesian. By (4.4) there exists a 2-simplex in X of the form



such that $p(g)$ is the degenerate edge on $v = p(x) = p(y)$ and g is an isomorphism in the fiber quasicategory $C = p^{-1}(v)$, and hence an isomorphism in the quasicategory X, since any morphism between quasicategories (such as $C \to X$) sends isomorphisms to isomorphisms.

Because p is an inner fibration between quasicategories, we have that since g is an isomorphism it is also p-Cartesian (3.1). Furthermore, in this context Cartesian edges are closed under composition (3.2), so since f is p-Cartesian we have that h is p-Cartesian, as desired. □

The following removes the quasicategory requirement from (5.1). It is proved as [2.4.2.8, (1) $\Longrightarrow$ (3)]. I do not understand the proof given there, so I give one here.

5.2. **Proposition.** *If $p\colon X \to S$ is a Cartesian fibration, then every locally p-Cartesian edge of X is p-Cartesian.*

Proof. Let $f\colon y \to z$ be a locally p-Cartesian edge of X. To show that f is a p-Cartesian edge, it suffices by (2.2) to show that for any basechange of p of the form $p'\colon X' \to \Delta^n$ with an edge f' of X' which is the pullback of f over $\langle n-1, n\rangle$, that the edge f' is p'-Cartesian. Such a pullback edge f' is certainly locally p'-Cartesian as such are preserved under base change, and since p' is a Cartesian fibration between quasicategories, it follows that f' is a p'-Cartesian edge by (5.1). □

Thus we have the following consequence.

5.3. **Theorem.** *If $p\colon X \to S$ is a Cartesian fibration, any edge f of X which is a fiber isomorphism is a p-Cartesian edge.*

Proof. Because p is a Cartesian fibration it is an inner fibration. Every fiber isomorphism of an inner fibration is locally p-Cartesian (4.3). Every locally p-Cartesian edge of a Cartesian fibration is p-Cartesian (5.2). □

5.4. **Corollary.** *If $p\colon X \to S$ is a Cartesian fibration, any degenerate edge of X is a Cartesian edge.*

6. Another proof

We here give another proof of (5.3).

A map $p\colon X \to S$ is a **locally Cartesian fibration** [2.4.2.6] if it is an inner fibration, and if the base change p' of p along any map $\Delta^1 \to S$ is a Cartesian fibration.

That is, locally Cartesian fibrations are $p\colon X \to Y$ such that

(1) p is an inner fibration, and
(2') for every edge $f\colon x \to y$ in S and vertex $\widetilde{y}$ in X such that $p(\widetilde{y}) = y$, there exists a *locally* p-Cartesian edge $\widetilde{f}\colon \widetilde{x} \to \widetilde{y}$ in X such that $p(\widetilde{f}) = f$.

In particular, any Cartesian fibration is a locally Cartesian fibration, since locally Cartesian edges are Cartesian edges.

We will need the following "composition criterion" for an edge in a locally Cartesian fibration to be a Cartesian edge, which is proved as [2.4.2.7, (3) $\Longrightarrow$ (1)]. I will give a proof of it below (§8).

6.1. **Proposition.** *Let $p\colon X \to S$ be a locally Cartesian fibration, and consider an edge $f\colon y \to z$ in X. Suppose f has the following property: for every 2-simplex σ in X of the form*

$$\begin{array}{ccccc} & & y & & \\ & {}^{g}\nearrow & & \searrow^{f} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

such that g is locally p-Cartesian, the edge h is also locally p-Cartesian. Then f is p-Cartesian.

Second proof of (5.3). Let $f\colon y \to z$ be an edge of X which is a fiber isomorphism of p. Since p is a Cartesian fibraton and thus a locally Cartesian fibration, to show that f is p-Cartesian we apply the criterion of of (6.1): for every 2-simplex σ in X of the form

$$\begin{array}{ccccc} & & y & & \\ & {}^{g}\nearrow & & \searrow^{f} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

such that g is locally p-Cartesian, we show that h is also locally p-Cartesian.

Because locally Cartesian edges are preserved and detected by base change, it suffices to consider $p'\colon X' \to \Delta^2$ obtained as the basechange of p along $\tau\colon \Delta^2 \to S$ representing $p(\sigma)$, together with the 2-simplex σ' of X' which is the pullback of σ over $\langle 012\rangle$. Write f', g', h' for the evident pullback edges of σ'.

We have that g' is locally p'-Cartesian as it is a pullback of a locally Cartesian edge. I claim that f' is also locally p'-Cartesian. To see this, observe that there is a commutative square

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{\langle 12\rangle} & \Delta^2 \\ {\scriptstyle\langle 00\rangle}\downarrow & & \downarrow{\scriptstyle\tau} \\ \Delta^0 & \xrightarrow[v]{} & S \end{array} \qquad \begin{array}{ccc} p''' & \Longrightarrow & p' \\ \Downarrow & & \Downarrow \\ p'' & \Longrightarrow & p \end{array} \qquad \begin{array}{ccc} f''' & \longmapsto & f' \\ \downarrow & & \downarrow \\ f'' & \longmapsto & f \end{array}$$

where v represents the vertex $p(y) = p(z)$. Condsider the basechanges p', p'', and p''' of p along the maps in the square, and let f', f'', and f''' be the pullback edges of f over $\langle 12\rangle$ in Δ^2, $\langle 00\rangle$ in Δ^0, and $\langle 01\rangle$ in Δ^1 respectively. Since f is a fiber isomorphism, the edge f' is an isomorphism in the domain of p'', and so is a p''-Cartesian edge since p'' is an inner fibration between quasicategories (3.1). Therefore its pullback f''' is a p'''-Cartesian edge and so locally p'''-Cartesian, from which it follows that f' is locally p'-Cartesian, since locally Cartesian edges are detected under base change.

Since p' is a base change of p, it is a Cartesian fibration, and being a Cartesian fibration between quasicategories it follows that the locally p'-Cartesian edge f' is in fact a p'-Cartesian edge (5.1). Furthermore, the edge g' is also p'-Cartesian by (4.1) since it is locally p'-Cartesian and $p'(g') = \langle 01\rangle$.

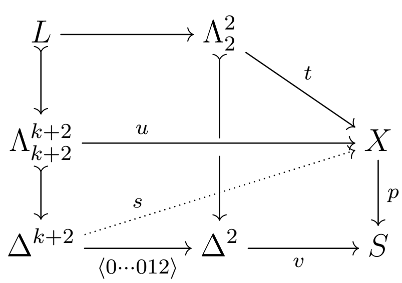
Again since p' is an inner fibration between quasicategories, we conclude that the "composite" h' of g' and f' is p'-Cartesian (3.2). Since locally Cartesian edges are detected by basechange we conclude that h is locally p-Cartesian. Thus we have verified the criterion of (6.1) to show that f is p-Cartesian. □

7. Criteria for Cartesian edges

We give explicit criteria for Cartesian edges in inner fibrations. The idea of the proof of the following is taken from the proof of [2.4.2.7] (specifically, from part of the proof that (3) $\Longrightarrow$ (1)).

7.1. **Proposition.** *Let $p\colon X \to S$ be an inner fibration, and let $f\colon y \to z$ be an edge of X. The following are equivalent.*

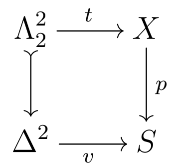
1. *f is a p-Cartesian edge.*
2. *The evident restriction map $\pi\colon X_{/f} \to X_{/z} \times_{S_{/pz}} S_{/pf}$ has contractible fibers.*
3. *For every $k \geq 0$ and every commutative diagram of the form*



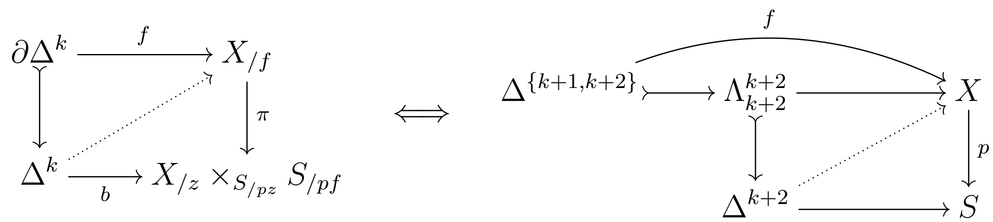
   such that $u(\langle k+1, k+2\rangle) = f$, where $L = \Delta^{\{0,\ldots,k,k+2\}} \cup \Delta^{\{k+1,k+2\}}$ is the preimage of $\Lambda^2_2 \subset \Delta^2$ under $\langle 0\cdots 012\rangle$, there exists a map $s\colon \Delta^{k+2} \to X$ such that $s|\Lambda^{k+2}_{k+2} = u$ and $ps = v\langle 0\cdots 012\rangle$.
4. *For every basechange $p'\colon X' \to \Delta^2$ of p along a map $v\colon \Delta^2 \to S$, and every edge $f'\colon y' \to z'$ of X' which is a pullback of f' over $\langle 12\rangle$, the evident restriction map*
$$\pi'\colon X'_{/f'} \to X'_{/z'} \times_{\Delta^2_{/\langle 2\rangle}} \Delta^2_{/\langle 12\rangle}$$
   has contractible fiber over any vertex of the codomain of π' which projects to $\langle 0\rangle \in \Delta^1 = \Delta^2_{/\langle 12\rangle}$.

Proof. Recall that by definition f is a p-Cartesian edge if and only if π is a trivial fibration. Because p is an inner fibration, π is a right fibration [2.1.2.1], and a right fibration is a trivial fibration if and only if it has contractible fibers [2.1.3.4]. This gives (1) $\Longleftrightarrow$ (2).

A vertex of $X_{/z} \times_{S_{/pz}} S_{/pf}$ has the form (h', v'), where $h' \in (X_{/z})_0$ and $v' \in (S_{/pf})_0$ correspond to maps $h\colon \Delta^1 \to X$ and $v\colon \Delta^2 \to S$ such that there is a commutative diagram



where $t|\Delta^{\{0,2\}} = h$ and $t|\Delta^{\{1,2\}} = f$. A straightforward argument involving the correspondence between lifting problems



shows that lifting problems on the left such that the image of b is a single vertex correspond exactly to the lifting problems of the type described in (3), thus proving (2) $\Longleftrightarrow$ (3).

The lifting problem of (3) is equivalent to a lifting problem for the basechange $p'\colon X' \to \Delta^2$ of p along v. A similar argument involving correspondence of lifting problems involving p' and π' gives (3) $\Longleftrightarrow$ (4). □

A a consequence we learn that the property of an edge in an inner fibration being Cartesian "lives" over 2-simplices, an observation which is implicit in [2.4.2] but apparently not stated explicitly there. (It can also be deduced using [2.4.4.3].)

7.2. **Corollary.** *If $p\colon X \to S$ is an inner fibration and f is an edge in X, then f is a p-Cartesian edge if and only if for every basechange $p'\colon X' \to \Delta^2$ of p along a map $v\colon \Delta^2 \to S$ with $v(\langle 12 \rangle) = p(f)$, the pullback edge f' of f in X' over $\langle 12 \rangle$ is p'-Cartesian.*

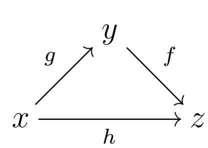
In fact (7.1) gives us something a little bit sharper than this, which will be used in the proof of (6.1) below.

8. Proof of the composition criterion for Cartesian edges in a locally Cartesian fibration (6.1)

The proof of the "composition criterion" (6.1) in [2.4.2.7] is quite terse. I give a more leisurely exposition here (which is just an elaboration of Lurie's proof, part of which is here contained in the proof of (7.1)).

Proof of (6.1). Consider a map $p\colon X \to S$ and an edge $f\colon y \to z$ in X. Assume we know the following

(a) p is a locally Cartesian fibration, and
(b) for every 2-simplex σ in X of the form



such that g is locally p-Cartesian, the edge h is also locally p-Cartesian.

We want to show that f is a p-Cartesian edge.

Let's say that an edge f in the domain of some p is "nice" if (a) and (b) hold. It is clear that if $p'\colon X' \to S'$ is a basechange of p, and if f' is an edge of X' which is a pullback of f, then f' is "nice" whenever f is "nice". This is because locally Cartesian fibrations are preserved under basechange, and locally Cartesian edges are preserved and detected by basechange.

To show that f is p-Cartesian we will apply the criterion of statement (4) of (7.1). In view of the remarks of the previous paragraph, it thus suffices to show that for any locally Cartesian fibration $p\colon X \to S$ with $S = \Delta^2$ and any "nice" edge $f\colon y \to z$ of X such that $pf = \langle 12\rangle$, the fiber of $\pi\colon X_{/f} \to X_{/z} \times_{\Delta^2} \Delta^{\{0,1\}}$ over any vertex which projects to $\langle 0\rangle \in \Delta^1$ is contractible. (Here I am implicitly using the unique isomorphisms $\Delta^2_{/\langle 2\rangle} = \Delta^2$ and $\Delta^2_{/\langle 12\rangle} = \Delta^{\{0,1\}}$.) Note that in this case p is a locally Cartesian fibration between quasicategories.

We introduce the following notation. For every map $q\colon K \to X$ we define Y_q and T_q by pullback squares

$$\begin{array}{ccccc} Y_q & \longrightarrow & T_q & \longrightarrow & \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ X_{/q} & \longrightarrow & \Delta^2_{/pq} & \longrightarrow & \Delta^2 \end{array}$$

where the maps in the bottom row are the evident restrictions along $\varnothing \to K \xrightarrow{q} X$. It is straightforward to check that $T_q \to \{0\}$ is always an isomorphism. Using this, one sees that for every $j\colon L \to K$ we have pullback squares

$$\begin{array}{ccccccc} Y_q & \longrightarrow & Y_{qj} & \longrightarrow & T_q & \xrightarrow{\sim} & \{0\} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_{/q} & \longrightarrow & X_{/qj} \times_{\Delta^2_{/pqj}} \Delta^2_{/pq} & \longrightarrow & \Delta^2_{/pq} & \longrightarrow & \Delta^2 \end{array}$$

where the maps are the evident ones.

In particular, taking q representing the edge f and $j\colon \{1\} \to \Delta^1$ we have a pullback square

$$\begin{array}{ccc} Y_f & \xrightarrow{\widetilde{\pi}} & Y_z \\ \downarrow & & \downarrow \\ X_{/f} & \xrightarrow[\pi]{} & X_{/z} \times_{\Delta^2} \Delta^{\{0,1\}} \end{array}$$

with the property that every vertex $(a', \langle 0\rangle)$ in $X_{/z} \times_{\Delta^2} \Delta^{\{0,1\}}$ is in the image of $Y_z \rightarrowtail X_{/z} \times_{\Delta^2} \Delta^{\{0,1\}}$. Thus, to show that the fibers of π over such vertices are contractible, it suffices to show that all fibers of $\widetilde{\pi}$ are contractible, and thus suffices to show that $\widetilde{\pi}$ is a trivial fibration. As noted earlier π is a right fibration, and therefore so is the basechange $\widetilde{\pi}$. Thus to show $\widetilde{\pi}$ is a trivial fibration, it suffices [2.4.2.4], [2.4.4.6] to show that $\widetilde{\pi}$ is a categorical equivalence.

Since p is a locally Cartesian fibration, there exists a locally p-Cartesian edge $g\colon x \to y$ in X with $p(g) = \langle 01\rangle$. Because X is a quasicategory we can extend along $\Lambda^2_1 \subset \Delta^2$ to produce

$\tau\colon \Delta^2 \to X$ representing a 2-simplex in X of the form

$$\begin{array}{ccccc} & & y & & \\ & \overset{g}{\nearrow} & & \overset{f}{\searrow} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

such that $p(\tau) = \langle 012 \rangle$. Our hypothesis that f is a "nice" edge implies that h is also a locally p-Cartesian edge. Because $p(g) = \langle 01 \rangle$ and $p(h) = \langle 02 \rangle$, these locally p-Cartesian edges are in fact p-Cartesian edges by (4.1).

We have the following commutative diagram of subcomplexes of Δ^2:

$$\begin{array}{ccccc} \Delta^2 & \hookleftarrow & \Lambda^2_1 & \hookleftarrow & \Delta^{\{1,2\}} \\ \uparrow & & & & \uparrow \\ \Lambda^2_0 & \hookleftarrow & \Delta^{\{0,2\}} & \hookleftarrow & \Delta^{\{2\}} \end{array}$$

inducing by restriction a commutative diagram

$$\begin{array}{ccccc} Y_\tau & \xrightarrow{\alpha_1} & Y_{\tau|\Lambda^2_1} & \xrightarrow{\gamma} & Y_f \\ {\scriptstyle \alpha_0}\downarrow & & & & \downarrow{\scriptstyle \widetilde{\pi}} \\ Y_{\tau|\Lambda^2_0} & \xrightarrow[\alpha_2]{} & Y_h & \xrightarrow[\delta]{} & Y_z \end{array}$$

I'll show that each of $\alpha_0, \alpha_1, \alpha_2, \gamma, \delta$ is a trivial fibration and thus a categorical equivalence, whence $\widetilde{\pi}$ is a categorical equivalence by the 2-out-of-3 property for categorical equivalences. This will complete the proof of the claim.

The maps $\alpha_0, \alpha_1, \alpha_2$ are basechanges of the evident restrictions $X_{/\tau} \to X_{/\tau|\Lambda^2_0} \times_{S_{/p\tau|\Lambda^2_0}} S_{/p\tau}$, $X_\tau \to X_{\tau|\Lambda^2_1} \times_{S_{/p\tau|\Lambda^2_1}} S_{/p\tau}$, and $X_{\tau|\Lambda^2_0} \to X_{/h} \times_{S_{/ph}} S_{/p\tau|\Lambda^2_0}$ respectively, each of which is a trivial fibration since p is an inner fibration and each of $\Lambda^2_0 \subset \Delta^2$, $\Lambda^2_1 \subset \Delta^2$, and $\Delta^{\{0,2\}} \subset \Lambda^2_0$ are left anodyne [2.1.2.5].

Since g and h are p-Cartesian edges, the map δ is a basechange of the trivial fibration $X_{/h} \to X_{/z} \times_{S_{/pz}} S_{/ph}$, while γ is a basechange of $X_{/\tau|\Lambda^2_1} \to X_{/f} \times_{S_{/pf}} S_{/p\tau|\Lambda^2_1}$, which is itself a basechange of the trivial fibration $X_{/g} \to X_{/y} \times_{S_{/py}} S_{/pg}$. Thus both δ and γ are trivial fibrations as desired.

□

References

[Cam19] Alexander Campbell, *A counterexample in quasicategory theory* (2019), available at `arXiv:1904.04965`.

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222. Special volume celebrating the 70th birthday of Professor Max Kelly.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

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