ETALE EXTENSIONS OF λ-RINGS

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Abstract. Given a λ-ring A and a formally etale morphism $f\colon A \to B$ of commutative rings, one may ask: What are the possible λ-ring strutures on B such that f is a map of λ-rings? We give the answer: Such a lifted λ-ring structure on B is determined uniquely by a compatible choice of lifts of the Adams operations ψ^p from A to B for all primes p which satisfy Frobenius congruences. In other words, to extend a λ-ring structure along a formally etale morphism, we need not be concerned about the "non-linear" part of the λ-ring structures in question.

Contents

1. Introduction

We assume the reader is familiar with λ rings. A ψ-ring B is a commutative ring equipped with "Adams operations", i.e., ring endomorphisms $\psi^p\colon B \to B$ for each prime p which pairwise commute. A ψ-ring is said to satisfy the Frobenius condition if each ψ^p is a lift of Frobenius on B/pB. Every λ-ring has an underlying ψ-ring, which necessarily satisfies the Frobenius condition.

Fix a λ-ring A and a map $f\colon A \to B$ of commutative rings which is formally etale. We will prove the following.

- If B is a ψ-ring satisfying the Frobenius condition, and $f\colon A \to B$ is a map of ψ-rings, then there exists a unique λ-ring structure on B compatible with the given ψ-ring structure, making f a map of λ-rings (5.4).
- If, in addition, $h\colon B \to C$ is a map of ψ-rings such that $hf\colon A \to C$ is a map of λ-rings, then h is a map of λ-rings (5.5).

We will prove these by first dealing with the analogous result in the case of θ^p-rings, which are a p-local analogue of λ-rings. We will then prove the result for λ-rings using a characterization of λ-rings in terms of their underlying θ^p-ring structures (4.10). This characterization is of independent interest; I am unaware that it has appeared in the literature up to now.[1]

2. θ^p-rings

2.1. θ^p **rings.** Fix a prime $p \in \mathbb{Z}$. A θ^p**-ring** is a commutative ring R equipped with a function $\theta^p\colon R \to R$ satisfying the following identities:

- $\theta^p(a+b) = \theta^p(a) + \theta^p(b) - C_p(a,b)$, where $C_p(x,y) = \sum_{i=1}^{p-1} p^{-1}\binom{p}{i} x^i y^{p-i} \in \mathbb{Z}[x,y]$;

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[1] I have a vague sense that I have run across it before somewhere, but I cannot find it.

- $\theta^p(ab) = \theta^p(a)b^p + a^p\theta^p(b) + p\theta^p(a)\theta^p(b)$;
- $\theta^p(1) = 0$.

A morphism of θ^p-rings is a ring homomorphism which commutes with θ^p. We write $\theta^p\mathrm{Ring}$ for the category of θ^p-rings.[2]

Note that for $n \in \mathbb{Z}$, the above identities for θ^p imply

- $\theta^p(na) = n\,\theta^p(a) - F_p(n)a^p$ where $F_p(n) = (n^p - n)/p \in \mathbb{Z}$.

(This has the amusing consequence that for p *odd* we have $\theta^p(-a) = -\theta^p(a)$, while on the other hand when $p = 2$ we have $\theta^2(-a) = -\theta^2(a) - a^2$.)

2.2. **θ^p-subrings.** Given any collection $\{a_s\}_{s\in S}$ of elements of a θ^p-ring R, and any polynomial $f(\underline{x}) \in \mathbb{Z}[x_s \,|\, s \in S]$ with integer coefficients with variables indexed by S, it is straightforward to see using the above identities that there exists a polynomial $g(\underline{x}, \underline{y}) \in \mathbb{Z}[x_s, y_s \,|\, s \in S]$ such that

$$\theta^p(f(a_s)) = g(a_s, \theta^p(a_s)).$$

This implies the following.

2.3. **Proposition.** *Let R be a θ^p-ring, and let $X \subseteq R$ be a subset closed under θ^p (i.e., such that $x \in X$ implies $\theta^p(x) \in X$). Then the subring $S \subseteq R$ generated as a ring by the set X is also a subset closed under θ^p, and thus is itself a θ^p-ring with θ^p-operation obtained by restriction from R.*

2.4. **θ^p-ideals.** An ideal $I \subseteq R$ in a θ^p-ring is a **θ^p-ideal** if for all $x \in I$ and $a \in R$, we have that $\theta^p(a + x) - \theta^p(a) \in I$. It is straightforward to show that an ideal $I \subseteq R$ is θ^p-ideal *if and only if* the quotient ring R/I inherits a (necessarily unique) θ^p-ring structure so that the quotient map $R \to R/I$ is a homomorphism of θ^p-rings.

2.5. **Proposition.** *Let R be a θ^p-ring, let $X \subseteq R$ be a subset, and let $I = (X)$ be the ideal in R generated by X. Then I is a θ^p-ideal if and only if $\theta^p(x) \in I$ for all $x \in X$.*

Proof. The "only if" part is clear, since $\theta^p(x) = \theta^p(0+x) - \theta^p(0)$. Suppose then that $X \subseteq R$ satisfies the given condition; we want to show that $I = (X)$ is a θ^p-ideal.

Let $J = \{\, x \in I \mid \theta^p(x) \in I \,\}$. Note that; (1) if $x, y \in J$, then $\theta^p(x+y) = \theta^p(x) + \theta^p(y) - C_p(x, y) \in I$ and thus $x + y \in J$; (2) if $x \in J$ and $a \in R$, then $\theta^p(ax) = \theta^p(a)x^p + a^p\theta^p(x) + p\theta^p(a)\theta^p(x) \in I$, and thus $ax \in J$. Thus J is a subideal of I containing X, whence $J = I$, and thus I is closed under the operation θ^p. Now note that if $x \in I$ and $a \in R$, we have that

$$\theta^p(a + x) - \theta^p(a) = \theta^p(x) - C_p(a, x) \in I,$$

as desired. $\square$

2.6. **Corollary.** *If R is a θ^p-ring and $I, J \subseteq R$ are θ^p-ideals, then $IJ \subseteq R$ is a θ^p-ideal.*

Proof. If $x \in I$ and $y \in J$, then $\theta^p(xy) = \theta^p(x)y^p + x^p\theta^p(y) + p\theta^p(x)\theta^p(y) \in IJ$, and thus IJ is a θ^p-ideal by (2.5). $\square$

2.7. **ψ^p-rings.** A **ψ^p-ring** is a pair (R, ψ^p) consisting of a commutative ring R equipped with a commutative ring map $\psi^p\colon R \to R$. There is an evident category $\psi^p\mathrm{Ring}$ of ψ^p-rings, whose morphisms are ring homomorphism which commute with ψ^p. (The "p" in the term "ψ^p-ring" and notation "ψ^p" and "$\psi^p\mathrm{Ring}$" is meant to stand for a prime p. In this paragraph it has served a purely decorative role, but it will matter in the following paragraph.)

We say that a ψ^p-ring (R, ψ^p) satisfies the **Frobenius condition** if ψ^p is a lift of Frobenius, i.e., if for all $a \in R$ we have that $\psi^p(a) \equiv a^p \mod pR$. We write $\psi^p\mathrm{Ring}_{\mathrm{Fr}}$ for the full subcategory of $\psi^p\mathrm{Ring}$ consisting of objects which satisfy the p-Frobenius condition.

[2]These are also known as δ-ring relative to p (Joyal [Joy85a]) or a ring with p-derivation (Buium). We are following Bousfield's [Bou96] terminology here.

2.8. **The Adams operation of a θ^p-ring.** Given a θ^p-ring R, we define the **Adams operation** $\psi^p = \psi^p\colon R \to R$ by $\psi^p(a) = a^p + p\theta^p(a)$. It is immediate that (R, ψ^p) is a ψ^p-ring, which furthermore satisfies the Frobenius condition. Thus, we have obtained a forgetful functor

$$\theta^p\mathrm{Ring} \to \psi^p\mathrm{Ring}_{\mathrm{Fr}} \subset \psi^p\mathrm{Ring}.$$

2.9. **The congruence criterion.** The following gives a complete criterion for constructing a θ^p-ring compatible with a given ψ^p-ring structure (R, ψ^p) the case that R has no p-torsion.

2.10. **Proposition.** *Let R be a ψ^p-ring satsifying the Frobenius condition. If R is also p-torsion free, then there exists a unique θ^p-ring structure on R, compatible with the given ψ^p.*

Proof. Given such (R, ψ^p), define a function $\theta^p\colon R \to R$ by $\theta^p(a) = (\psi^p(a) - a^p)/p$, and verify the identities for a θ^p-ring directly. ☐

This implies that the forgetful functor $\mathcal{U}\colon \theta^p\mathrm{Ring} \to \psi^p\mathrm{Ring}_{\mathrm{Fr}} \subseteq \psi^p\mathrm{Ring}$ restricts to an equivalence of categories

$$\theta^p\mathrm{Ring}_{\mathrm{tf}} \xrightarrow{\sim} (\psi^p\mathrm{Ring}_{\mathrm{Fr}})_{\mathrm{tf}},$$

where these denote full subcategories of p-torsion free objects in $\theta^p\mathrm{Ring}$ and $\psi^p\mathrm{Ring}_{\mathrm{Fr}}$ respectively.

2.11. **θ^p-rings as coalgebras.** Given a ring R, we define a ring $V(R) = V_p(R)$ as follows. The underlying set of $V(R)$ is $R \times R$, and addition and multiplication are defined by

$$(x, y) + (x', y') := (x + x', y + y' - C_p(x, x')),$$
$$(x, y) \cdot (x', y') := (xx', yx'^p + x^p y' + p\, yy').$$

The map $\epsilon_V\colon V(R) \to R$ defined by $\epsilon_V(x, y) = x$ is a ring homomorphism.

A **V-coalgebra** is a pair (R, α) consisting of a ring R and a ring homomorphism $\alpha\colon R \to V(R)$ such that $\epsilon_V \alpha = \mathrm{id}_R$. A map of V-coalgebras is a ring homomorphism which commutes with α.

2.12. **Proposition.** *There is an equivalence of categories $\theta^p\mathrm{Ring} \xrightarrow{\sim} V\mathrm{Coalg}$, which sends the θ^p-ring (R, θ^p) to the V-coalgebra (R, α), where $\alpha(x) = (x, \theta^p(x))$.*

Proof. Straightforward. ☐

2.13. *Remark.* One may also describe θ^p-rings as the coalgebras for a certain comonad W on Ring. The comonad W is in fact the *cofree comonad* on the augmented endofunctor (V, ϵ_V). It is well-known that the underlying functor $W\colon \mathrm{Ring} \to \mathrm{Ring}$ of this comonad is in fact the p-typical Witt functor.[3]

2.14. **ψ^p-rings as coalgebras.** Similarly as above, let $G(R) = R \times R$ as a ring, and define $\epsilon_G\colon G(R) \to R$ by $\epsilon_G(x, y) = x$. A **G-coalgebra** is a pair (R, β) with $\beta\colon R \to G(R)$ a ring homomorphism such that $\epsilon_G \beta = \mathrm{id}_R$. Let $\pi\colon V(R) \to G(R)$ denote the map $\pi(x, y) = (x, x^p + py)$. It is clear that π is a ring homomorphism natural in R, and that $\epsilon_G \pi = \epsilon_V$.

2.15. **Proposition.** *There is an equivalence of categories $\psi^p\mathrm{Ring} \xrightarrow{\sim} G\mathrm{Coalg}$ which sends (R, ψ^p) to the G-coalgebra $(R, (\mathrm{id}, \psi^p))$. With respect to this equivalence and that of* (2.12)*, the forgetful functor $\theta^p\mathrm{Ring} \to \psi^p\mathrm{Ring}$ corresponds to the functor $V\mathrm{Coalg} \to G\mathrm{Coalg}$ which sends (R, α) to $(R, \pi\alpha)$.*

Proof. Immediate. ☐

[3] I believe this is an observation of Joyal [Joy85a].

2.16. **Limits and colimits of θ^p-rings.**

2.17. **Proposition.** *The category $\theta^p\mathrm{Ring}$ of θ^p-rings has all small limits and colimits, and the forgetful functor $\theta^p\mathrm{Ring} \to \mathrm{Ring}$ which sends a θ^p-ring to its underlying commutative ring preserves limits and colimits.*

Proof. To prove the statement about limits, note that if $A\colon \mathcal{C} \to \theta^p\mathrm{Ring}$ is a functor from a small category, and $U\colon \theta^p\mathrm{Ring} \to \mathrm{Ring}$ denotes the underlying ring functor, we can define a operator θ^p on the set $\lim_{\mathcal{C}} UA$ componentwise, and check that it satisfies the axioms for a θ^p-ring. It is then straightforward to check that this realizes the limit of $A\colon \mathcal{C} \to \theta^p\mathrm{Ring}$.

To prove statement about colimits, let $A\colon \mathcal{C} \to \theta^p\mathrm{Ring}$ be a functor from a small category, and $U\colon \theta^p\mathrm{Ring} \to \mathrm{Ring}$ the underlying ring functor. Let $S = \operatorname{colim}_{\mathcal{C}} UA$, the colimit of the diagram in Ring, with $i(c)\colon UA(c) \to S$ denoting the tautological maps. Each θ^p-ring $A(c)$ corresponds to a ring homomorphism $\alpha(c)\colon UA(c) \to V(UA(c))$ which is a section $\pi\colon V(UA(c)) \to UA(c)$, and these maps fit together to give a ring homomorphism

$$\alpha\colon S = \operatorname{colim}_{\mathcal{C}} UA \xrightarrow{\operatorname{colim}_{\mathcal{C}} \alpha} \operatorname{colim}_{\mathcal{C}} V(UA) \xrightarrow{(V(i(c)))} V(S),$$

and that $\pi\alpha = \mathrm{id}$. Thus (S, α) defines a θ^p-ring structure on S, and it is straightforward to verify that S is the colimit of A. $\square$

3. Lifting problems for θ^p-rings

3.1. **Lifting an ψ^p-ring structure to a θ^p-ring structure.** Given a ψ^p-ring (R, ψ^p), a **θ^p-ring structure** on it is a θ^p-ring structure on R such that $\psi^p(x) = x^p + p\theta^p(x)$.

We now consider the following problem: given a ψ^p-ring (R, ψ^p), what are the possible θ^p-ring structures on it? In view of (2.12) and (2.15), we see that a θ^p-ring structure on (R, ψ^p) corresponds exactly to a homomorphism α fitting in the diagram

$$\begin{tikzcd} & V(R) \arrow[d, "\pi"] \\ R \arrow[ur, dotted, "\alpha"] \arrow[r, "(\mathrm{id},\psi^p)"'] & R \times R \end{tikzcd}$$

Thus, to understand this lifting problem we must examine the homomorphism π.

Let $\overline{V}(R) = \pi(V(R)) \subseteq R \times R$ denote the image of π. It is a subring of $R \times R$, described as the subset $\{\, (x, y) \mid x^p \equiv y \mod p \,\}$. Let $I(R) = \operatorname{Ker}(\pi) = \operatorname{Ker}(V(R) \twoheadrightarrow \overline{V}(R)) \subseteq V(R)$.

3.2. **Proposition.** *Let R be a commutative ring, and let $\overline{V}(R)$ and $I(R)$ be defined as above.*

1. *As an abelian group, the quotient group $(R \times R)/\overline{V}(R)$ is isomorphic to R/pR.*
2. *As an abelian group, $I(R)$ is isomorphic to $R[p] = \operatorname{Ker}[p\colon R \to R]$, the additive group of p-torsion elements in R.*
3. *The ideal $I(R) \subseteq V(R)$ is square-zero, and thus*
$$0 \to I(R) \to V(R) \twoheadrightarrow \overline{V}(R) \subseteq R \times R$$
presents $V(R)$ as a square-zero extension of $\overline{V}(R)$ by $I(R)$.
4. *Let $\phi\colon \overline{V}(R) \to R/pR$ be the ring homomorphism defined by $\phi(x, y) = x^p = y$. As a $\overline{V}(R)$-module, $I(R)$ is isomorphic to $\phi^*(R[p])$, the module obtained by restricting scalars along ϕ from the evident R/pR-module structure on $R[p]$.*

Proof. The ring $\overline{V}(R)$ is isomorphic to the limit of the diagram

$$R \xrightarrow{x \mapsto x^p} R/pR \xleftarrow{y \leftarrow\!\shortmid y} R,$$

and (1) follows immediately.

Note that as a set, $I(R) = \{ (0,z) \mid pz = 0 \} \subseteq V(R)$. Statements (2) and (3) are immediate from the description of the ring structure on $V(R)$.

For statement (4), let $(x,y) \in \overline{V}(R)$, and choose any lift $(x,u) \in V(R)$, so that $pu = y - x^p$. Then for $(0,z) \in I(R)$, we have

$$(x,u) \cdot (0,z) = (0, x^p z + puz) = (0, x^p z) = (0, yz).$$

This verifies the claim about the module structure on $I(R)$. ☐

Thus, to construct a θ^p-ring structure on (R, ψ^p), we must (1) show that the image of $(\mathrm{id}, \psi^p)\colon R \to R \times R$ lies in $\overline{V}(R)$, and (2) lift the resulting map $R \to \overline{V}(R)$ to a homomorphism $\alpha\colon R \to V(R)$. Step (1) exactly says that ψ^p must satisfy the Frobenious condition.

3.3. *Remark.* If R has no p-torsion, then $I(R) = 0$, and thus $V(R) \to \overline{V}(R)$ is an isomorphism. Thus we recover the congruence criterion (2.10) for torsion free θ^p-rings.

3.4. **θ^p-ring structures and p-localization.** Fix a ψ^p-ring (R, ψ^p). Tensoring with $\mathbb{Z}_{(p)}$ gives rise to a ψ^p-ring $(R_{(p)}, \psi^p)$.

3.5. **Proposition.** *There is a one-to-one correspondence*

$$\left\{\theta^p\text{-ring structures on } (R, \psi^p)\right\} \longleftrightarrow \left\{\theta^p\text{-ring structures on } (R_{(p)}, \psi^p)\right\}$$

Proof. From (3.2), we have natural exact sequences of abelian groups

$$0 \to R[p] \to V(R) \xrightarrow{\pi} R \times R \to R/pR \to 0.$$

When we plug in the homomorphism $j\colon R \to R_{(p)}$, and observe that $R[p] \xrightarrow{\sim} R_{(p)}[p]$ and $R/pR \xrightarrow{\sim} R_{(p)}/pR_{(p)}$, we obtain a pullback square of rings of the form

$$\begin{tikzcd} V(R) \arrow[r, "V(j)"] \arrow[d, "\pi"'] & V(R_{(p)}) \arrow[d, "\pi"] \\ R \times R \arrow[r, "j\times j"'] & R_{(p)} \times R_{(p)} \end{tikzcd}$$

Also, $V(R_{(p)})$ is a $\mathbb{Z}_{(p)}$-module, as it fits in a finite exact sequence whose other terms are $\mathbb{Z}_{(p)}$-modules.

To define the correspondence asserted by the proposition, we observe that there is a bijection

$$\{ \alpha\colon R \to V(R) \mid \pi\alpha = (\mathrm{id}, \psi^p) \} \longleftrightarrow \{ \alpha'\colon R_{(p)} \to V(R_{(p)}) \mid \pi\alpha' = (\mathrm{id}, \psi^p) \},$$

sending α to the unique homomorphism α' such that $\alpha' j = V(j)\alpha$. ☐

Thus, the problem of lifting a ψ^p-structure to a θ^p-structure is (unsurprisingly) a purely p-local problem.

3.6. **The relative lifting problem for θ^p-rings.** Now we consider the following problem. Suppose we are given a θ^p-ring (A, θ^p) and a ψ^p-ring (B, ψ^p) which satisfies the Frobenius condition, together with homomophism $f\colon A \to B$ of ψ^p-rings (using the underlying ψ^p-structure of the (A, θ^p)). What are the possible θ^p-ring structures on (B, ψ^p) making f a map of θ^p-rings? In view of the previous sections, we see that providing such a structure amounts to producing a dotted arrow α_B in

$$\begin{tikzcd} A \arrow[r, "V(f)\alpha_A"] \arrow[d, "f"'] & V(B) \arrow[d, "\pi"] & \\ B \arrow[r, "(\mathrm{id},\psi^p)"'] \arrow[ur, dotted, "\alpha_B"] & \overline{V}(B) \arrow[r, phantom, "\subseteq"] & B \times B \end{tikzcd}$$

making the diagram commute.

Recall that $f\colon A \to B$ is **formally etale** if for every ring R and square-zero ideal $I \subseteq R$, and every commutative diagram of ring homomorphisms of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & R \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & R/I \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute.

3.7. **Proposition.** *If (A, θ^p) is a θ^p-ring, (B, ψ^p) a ψ^p-ring satisfying the Frobenius condition, and $f\colon A \to B$ is a map of ψ^p-rings which is formally etale as a map of commutative rings, then there exists a unique θ^p-ring structure on (B, ψ^p) making f a map of θ^p-rings.*

Proof. Immediate in view of the above remarks and (3.2)(3). ☐

The lift of the previous proposition is natural.

3.8. **Proposition.** *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B \arrow[ur, "h"'] & \end{tikzcd}$$

of ring maps, such that (i) A, B, and C are θ^p-rings, (ii) f and g are maps of θ^p-rings, (iii) h is a map of ψ^p-rings, and (iv) f is formally etale. Then h is a map of θ^p-rings.

Proof. Consider the diagrams

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, "V(g)\alpha_A"] \arrow[d, "g"'] & V(C) \arrow[d, "\pi"] \\ B \arrow[r, "h"'] & C \arrow[r, "(\mathrm{id}, \psi^p_C)"'] \arrow[ur, "\alpha_C"] & C \times C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "V(f)\alpha_A"] \arrow[d, "f"'] & V(B) \arrow[r, "V(h)"] \arrow[d, "\pi"'] & V(C) \arrow[d, "\pi"] \\ B \arrow[r, "(\mathrm{id}, \psi^p_B)"'] \arrow[ur, "\alpha_B"] & B \times B \arrow[r, "h \times h"'] & C \times C \end{tikzcd}$$

The solid arrow diagrams commute, using that f and g are maps of θ^p-rings. The two outer rectangles are actually identical, since

$$V(g)\alpha_A = V(h)V(f)\alpha_A \qquad \text{and} \qquad (\mathrm{id}, \psi^p_C)h = (h, \psi^p_C h) = (h, h\psi^p_B) = (h \times h)(\mathrm{id}, \psi^p_B).$$

Thus the homomorphisms $\alpha_C h, V(h)\alpha_B\colon B \to V(C)$ must coincide, since f is formally etale and they are solutions to the same lifting problem, whence h is a map of θ^p-rings as desired. ☐

Given a θ^p-ring A, let $\theta^p\mathrm{Ring}(A)$ denote the category of θ^p-rings under A, and $\psi^p\mathrm{Ring}_{\mathrm{Fr}}(A)$ the category of ψ^p-rings under A which satisfy the Frobenius congruence. Let $\theta^p\mathrm{Ring}(A)_{\mathrm{f.etale}}$ and $\psi^p\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ denote the respective full subcategories consisting of objects $f\colon A \to B$ such that f is formally etale.

3.9. **Proposition.** *The evident forgetful functor $\theta^p\mathrm{Ring}(A)_{\mathrm{f.etale}} \to \psi^p\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ is an equivalence of categories.*

Proof. Clear from (3.7) and (3.8). ☐

The statement of the previous proposition remains true if we replace “formally etale” with any subclass of maps, such as “etale” or “weakly etale”.

3.10. *Remark.* In fact, we can generalize the above a little bit. Say that $f\colon A \to B$ is **p-formally etale** if $f \otimes \mathbb{Z}_{(p)}\colon A_{(p)} \to B_{(p)}$ is formally etale. In view of (3.5), the propositions (3.7), (3.8), and (3.9) apply with “formally etale” replaced by “p-formally etale”.

4. λ-rings

A **λ-ring** is a commutative ring R equipped with functions $\lambda^n\colon R \to R$ for $n \geq 0$, satisfying identities of the form

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$,
- $\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$,
- $\lambda^n(xy) = P_n(\lambda^1(x), \dots, \lambda^n(x); \lambda^1(y), \dots, \lambda^n(y))$,
- $\lambda^m\lambda^n(x) = P_{m,n}(\lambda^1(x), \dots, \lambda^{mn}(x))$,

where P_n and $P_{m,n}$ are certain polynomials with integer coefficients. We refer the reader elsewhere for a complete definition, for instance [Yau10]. We write $\lambda\mathrm{Ring}$ for the category of λ-rings.

The purpose of this section is to give a characterization of λ-rings in terms of θ^p-rings. That is, we will show that a λ-ring is nothing more than a commutative ring R equipped with θ^p-ring structures for each prime p, which are compatible in the sense that

$$\psi^p\theta^q = \theta^q\psi^p \qquad \text{for all distinct primes } p, q.$$

4.1. *Remark.* Joyal proved [Joy85b] that the data of a λ-ring structure on R is equivalent to: a θ^p-structure on R for each prime, together with for each pair of distinct primes p, q a somewhat non-trivial relation relating $\theta^p\theta^q(x)$ and $\theta^q\theta^p(x)$ up to terms which do not involve compositions of θ-operations. (See also [Bor11, §1.19].) Our characterization allows us to avoid explicit mention of this relation, replacing it with the simpler one above.

4.2. **Facts about lambda rings.** We note the following facts about λ-rings.

(1) λ-rings are a variety of universal algebra, and thus $\lambda\mathrm{Ring}$ is a locally presentable category. In particular, any functor $U\colon \lambda\mathrm{Ring} \to \mathcal{C}$ which preserves small colimits admits a right adjoint.
(2) Limits and colimits in $\lambda\mathrm{Ring}$ exist, and the evident forgetful functor $\lambda\mathrm{Ring} \to \mathrm{Ring}$ preserves limits and colimits.
(3) The free λ-ring on one generator F has the form $F \approx \mathbb{Z}[\lambda^n(x) \mid n \geq 1]$, where $x = \lambda^1(x)$ is the generator. In particular, it is torsion free as an abelian group.
(4) Any λ-ring R has natural Adams operations $\psi^n\colon R \to R$ for $n \geq 1$, which are ring homomorphisms; furthermore, $\psi^m\psi^n = \psi^{mn}$ and $\psi^1 = \mathrm{id}$.
(5) For p prime, $\psi^p(x) \equiv x^p \mod pR$ in any λ-ring R.
(6) For every prime p there exist natural functions $\theta^p\colon R \to R$ on any λ-ring such that $\psi^p(x) = x^p + p\theta^p(x)$, and (R, θ^p) is in fact a θ^p-ring, as can be shown by checking the appropriate formulas in the free λ-ring on one generator F, which is torsion free.
(7) For distinct primes p and q, we have that $\psi^p\theta^q = \theta^q\psi^p$ as functions on any λ-ring, as can be shown by verifying that $\psi^p\theta^q(x) = \theta^q\psi^p(x)$ in F.
(8) Let $\delta\colon F \to \mathbb{Z}$ be the λ-ring homomorphism from the free λ-ring on one generator sending the generator x to 0, and let $J = \operatorname{Ker}\delta$. Then J/J^2 is a free abelian group on $\{\lambda^n(x)\}_{n\geq 1}$.
(9) We have that $\psi^n(x) \equiv (-1)^{n-1}n\lambda^n(x) \mod J^2$. Thus, any sequence $\{u_n\}_{n\geq 0}$ of elements in J such that $nu_n \equiv \pm\psi^n(x) \mod J^2$ is a basis for J/J^2.

4.3. **Θ-rings.** A **Θ-ring** is the data $(R, \{\theta^p\})$ consisting of a commutative ring R and a choice for each prime $p \in \mathbb{Z}$ of a θ^p-structure on R, such that for all distinct primes p and q, we have that

$$\psi^p\theta^q = \theta^q\psi^p,$$

where $\psi^p(x) = x^p + p\theta^p(x)$ is the Adams operation associated to θ^p. We note that it is also the case that $\psi^p\theta^p = \theta^p\psi^p$, as this is true in any θ^p-ring.

A morphism $A \to B$ of Θ-rings is a map which commutes with all the structure, i.e., a ring homomorphism $f\colon A \to B$ such that $f\theta^p = \theta^p f$ for all p. We write $\Theta\mathrm{Ring}$ for the category of Θ-rings.

An ideal $I \subseteq R$ of a Θ-ring is a **Θ-ideal** if it is a θ^p-ideal for all p. It is clear that if I is a Θ-ideal, then R/I admits a unique Θ-ring structure as a quotient of the structure on R.

4.4. **Facts about subrings of Θ-rings.** We collect some facts for use in the proof in the next section.

Let R be a Θ-ring, and consider an ordinary subring $S \subseteq R$. Write $\Theta S \subseteq R$ for the ordinary subring generated by the set $S \cup \bigcup_p \theta^p(S)$. It is straightforward to show (see §2.2) that if S is generated as a subring by a subset $X \subseteq R$, then ΘS is generated as a subring by the subset $X \cup \{\, \theta^p(x) \mid x \in S,\ p \text{ prime}\,\}$.

It is clear from (2.3) that if $S \subseteq R$ is a subring, then $\bigcup_k \Theta^k S$ is the Θ-subring in R generated by S (i.e., the smallest subring of R containing S and closed under the θ^p operations).

Although a subring is not generally an ideal, it is a subgroup, and so it makes sense to talk about congruence modulo a subring: we say $x \equiv y \in S$ if $x - y \in S$, when $S \subseteq R$ is a subring and $x, y \in R$.

4.5. **Proposition.** *Let R be a Θ-ring, $S \subseteq R$ an ordinary subring, and $x \in S$. Then for all primes p and q, we have that*

$$\theta^p\theta^q(x) \equiv \theta^q\theta^p(x) \mod \Theta S.$$

Proof. If $p = q$ this is obvious. For distinct primes p and q we have

$$\psi^p\theta^q(x) = \theta^q(x)^p + p\,\theta^p\theta^q(x) \equiv p\,\theta^p\theta^q(x) \mod \Theta S,$$

and

$$\begin{aligned}\theta^q\psi^p(x) &= \theta^q(x^p + p\theta^p(x)) = \theta^q(x^p) + \big(p\theta^q\theta^p(x) - F_q(p)\theta^p(x)^q\big) - C_p(x^p, p\theta^p(x)) \\ &\equiv p\theta^q\theta^p(x) \mod \Theta S.\end{aligned}$$

Therefore $p(\theta^p\theta^q(x) - \theta^q\theta^p(x)) \in \Theta S$. By symmetry we also have $q(\theta^p\theta^q(x) - \theta^q\theta^p(x)) \in \Theta S$, and since p and q are relatively prime, a suitable integer combination of these congruences gives $\theta^p\theta^q(x) - \theta^q\theta^p(x) \in \Theta S$, as desired. $\square$

4.6. *Remark.* The argument of (4.5) actually shows that

$$\theta^p\theta^q(x) - \theta^q\theta^p(x) = f(x, \theta^p(x), \theta^q(x))$$

where f is some polynomial with integer coefficients. It is not hard to describe this polynomial explicitly, e.g., [Joy85b], [BS09, Def. 2.2] or [Bor11, (1.19.4)].

Let $\mathcal{P}$ denote the set of finite sequences $P = (p_1, \dots, p_k)$ of primes $p_i \in \mathbb{Z}$, including the empty sequence, and write $|P| = k$ for the length of the sequence P. For $x \in R$ and $P \in \mathcal{P}$ write $\theta^P(x) = \theta^{p_1} \cdots \theta^{p_k}(x)$.

4.7. **Proposition.** *Let R be a Θ-ring and $S \subseteq R$ ordinary subring. If $x, y \in \Theta S$ are such that $x \equiv y$ mod S, then $\theta^P(x) \equiv \theta^P(y) \mod \Theta^{|P|}S$ for all $P \in \mathcal{P}$.*

Proof. If $|P| = 0$, there is nothing to prove. If $P = (p)$, then if $x = y + a$ with $a \in S$, we have that $\theta^p(x) - \theta^p(y) = \theta^p(a) - C_p(y, a) \in \Theta S$.

The case of $|P| > 1$ is handled by induction on the length: given $\theta^P(x) \equiv \theta^P(y) \mod \Theta^{|P|}S$ with $\theta^P(x), \theta^P(y) \in \Theta^{|P|+1}S$, it follows using the length-one case already proved that $\theta^q\theta^P(x) \equiv \theta^q\theta^P(y)$ mod $\Theta^{|P|+1}S$ and $\theta^q\theta^P(x), \theta^q\theta^P(x) \in \Theta^{|P|+2}S$. $\square$

4.8. **Proposition.** *Let R be a Θ-ring, $S \subseteq R$ an ordinary subring, and $x \in S$. Let $P, Q \in \mathcal{P}$ be two sequences of the same length, where Q is obtained from P by reordering its elements. Then*

$$\theta^P(x) \equiv \theta^Q(x) \mod \Theta^{|P|-1}S.$$

Proof. It suffices to consider the case of pairs of sequences obtained by reordering an adjacent pair of elements. Thus, let p, q be primes, $U, V \in \mathcal{P}$ and let $P = (U, p, q, V)$ and $Q = (U, q, p, V)$. Then $\theta^V(x) \in \Theta^{|V|}S$, whence

$$\theta^p\theta^q\theta^V(x) \equiv \theta^q\theta^p\theta^V(x) \mod \Theta^{|V|+1}S$$

by (4.5). As $\theta^p\theta^q\theta^V(x), \theta^q\theta^p\theta^V(x) \in \Theta^{|V|+2}S$, it follows that

$$\Theta^U\theta^p\theta^q\Theta^V(x) \equiv \theta^U\theta^q\theta^p\theta^V(x) \mod \Theta^{|U|+1+|V|}S = \Theta^{|P|-1}S$$

by (4.7). $\square$

4.9. **Equivalence of Θ-rings and λ-rings.** There is an evident "forgetful" functor $U\colon \lambda\mathrm{Ring} \to \Theta\mathrm{Ring}$, by §4.2(6), (7).

4.10. **Proposition.** *The forgetful functor $U\colon \lambda\mathrm{Ring} \to \Theta\mathrm{Ring}$ is an equivalence of categories.*

Proof. Let R be a Θ-ring.

By (2.3), it is clear that for any $x \in R$, the subring $S \subseteq R$ generated as a ring by the set $\{\theta^P(x)\}_{P\in\mathcal{P}}$ is a sub-Θ-ring. We may thus construct the free Θ-ring on one generator E as the quotient of the polynomial ring $\mathbb{Z}[\theta^P(x) \mid P \in \mathcal{P}]$ by the ideal consisting of relations which hold in every Θ-ring.

Let $\mathcal{P}_\leq \subseteq \mathcal{P}$ denote the subset consisting of sequences $P = (p_1, \dots, p_k)$ such that $p_1 \leq \cdots \leq p_k$; note the evident bijection $\mathcal{P}_\leq \xrightarrow{\sim} \mathbb{Z}_{>0}$ sending $P \mapsto n(P) := p_1 \cdots p_k$. By inductive application of (4.8), we see that E is in fact generated as a commutative ring by the set $\{\theta^P(x)\}_{P\in\mathcal{P}_\leq}$. That is, if we write S_k for the subring of E generated by $\{\, \theta^P(x) \mid P \in \mathcal{P}_\leq,\ |P| \leq k \,\}$, we show by induction on k that $\Theta^k S_0 \subseteq S_k$, using (4.8). Thus $E = \bigcup_k \Theta^k S_0 = \bigcup_k S_k$.

Let $\delta\colon E \to \mathbb{Z}$ be the Θ-homomorphism sending $x \mapsto 0$, and let $I = \operatorname{Ker}\delta$. Clearly I is a Θ-ideal, and thus I^2 is a Θ-ideal by (2.6). Furthermore, as an ideal I is generated by the set $\{\theta^P(x)\}_{P\in\mathcal{P}_\leq}$, since these elements generate E as a ring and are contained in I. Therefore I/I^2 is generated as an abelian group by the $\theta^P(x)$ with $P \in \mathcal{P}_\leq$. Note also that $p\theta^p(x) \equiv \psi^p(x) \mod I^2$, and thus that $n\theta^P(x) \equiv \psi^{n(P)}(x) \mod I^2$.

Now let F denote the free λ-ring on one generator, with augmentation ideal J, and let $\phi\colon E \to U(F)$ be the Θ-ring map sending generator to generator. By §4.2(3), (8), (9), the induced map

$$I/I^2 \to J/J^2 \approx \mathbb{Z}\{\text{“}n^{-1}\psi^n(x)\text{”}\}_{n\geq 1} \approx \mathbb{Z}\{\text{“}\lambda^n(x)\text{”}\}_{n\geq 1}$$

sends the generators $\theta^P(x)$ of I/I^2 with $P \in \mathcal{P}_\leq$ bijectively to a basis of J/J^2, whence $I/I^2 \to J/J^2$ is a bijection. The composite of

$$\mathbb{Z}[\theta^P(x) \mid P \in \mathcal{P}_\leq] \twoheadrightarrow E \xrightarrow{\phi} F \approx \mathbb{Z}[\lambda^n(x) \mid n \geq 1]$$

is a homomorphism of polynomial rings sending $\theta^P(x)$ to $\pm\lambda^{n(P)}(x) \bmod J^2$, and thus is a bijection. Therefore ϕ is an isomorphism as desired.

The forgetful functor $U\colon \lambda\mathrm{Ring} \to \Theta\mathrm{Ring}$ preserves small colimits (since in both $\lambda\mathrm{Ring}$ and $\Theta\mathrm{Ring}$ colimits are computed as in commutative rings, by §4.2(2) and (2.17) respectively), and thus U admits a right adjoint $G\colon \Theta\mathrm{Ring} \to \lambda\mathrm{Ring}$ by §4.2(1).

Now let $\eta\colon \mathrm{Id} \to GU$ and $\epsilon\colon UG \to \mathrm{Id}$ be unit and counit of the $U \dashv G$ adjunction. For a λ-ring R, the composite

$$R \approx \lambda\mathrm{Ring}(F, R) \xrightarrow{U} \Theta\mathrm{Ring}(UF, UR) \xrightarrow{\phi^*} \Theta\mathrm{Ring}(E, UR) \approx R$$

is the identity function on the set R, and thus $U\colon \lambda\mathrm{Ring}(F, R) \to \Theta\mathrm{Ring}(UF, UR)$ is a bijection for every λ-ring R. The composite map

$$R \approx \lambda\mathrm{Ring}(F, R) \xrightarrow{U} \Theta\mathrm{Ring}(UF, UR) \approx \lambda\mathrm{Ring}(F, GUR) \approx GUR \tag{4.11}$$

is therefore a bijection, whence $\eta\colon \mathrm{Id} \to GU$ is an isomorphism. For a Θ-ring S, the composite map

$$S \approx \Theta\mathrm{Ring}(E, S) \xrightarrow{(\phi^*)^{-1}} \Theta\mathrm{Ring}(UF, S) \approx \lambda\mathrm{Ring}(F, GS) \xrightarrow[\sim]{\phi^* \circ U} \Theta\mathrm{Ring}(E, UGS) \approx UGS \tag{4.12}$$

is a bijection of sets (using that (4.11) is always an isomorphism), and it is straightforward to check that the composite of (4.12) is an inverse to $\epsilon\colon UGS \to S$. Thus we have shown that $U \dashv G$ is an equivalence of categories. □

5. Lifting problems for λ-rings

In view of (4.10), we will treat λ-rings as synonymous with Θ-rings.

5.1. **Ψ-rings.** A **Ψ-ring** is the data $(R, \{\psi^p\})$ consisting of a commutative ring R and a choice for each prime $p \in \mathbb{Z}$ of a ψ^p-ring structure on R, such that for all distinct primes p and q we have that $\psi^p\psi^q = \psi^q\psi^p$. We write $\Psi\mathrm{Ring}$ for the category of Ψ-rings.

We say that a Ψ-ring satisfies the **Frobenius condition** if each ψ^p does, i.e., if $\psi^p(x) \equiv x^p \mod p$ for all p. We write $\Psi\mathrm{Ring}_{\mathrm{Fr}} \subseteq \Psi\mathrm{Ring}$ for the full subcategory of Ψ-rings satisfying the Frobenius condition. There is an evident forgetful functor

$$\lambda\mathrm{Ring} \to \Psi\mathrm{Ring}_{\mathrm{Fr}} \subseteq \Psi\mathrm{Ring}.$$

We observe that if $F\colon \mathrm{Ring} \to \mathrm{Ring}$ is any functor and R is a Ψ-ring, then $F(R)$ inherits a natural Ψ-ring structure, with operations $\psi^p_{F(R)} = F(\psi^p_R)$. For instance, this applies to the functors $V_p\colon \mathrm{Ring} \to \mathrm{Ring}$ of §2.11.

5.2. **Lifting a Ψ-ring structure to a λ-ring structure.** We now consider the following problem: given a Ψ-ring R, what are the possible λ-ring structures on R compatible with the given Ψ-ring structure? In view of (4.10), (2.12), and (2.15), such a λ-ring structure on R corresponds exactly to a choice for each prime p of Ψ-ring homomorphism α_p fitting in the commutative diagram

$$\begin{tikzcd} & V_p(R) \arrow[d, "\pi"] \\ R \arrow[ur, dotted, "\alpha_p"] \arrow[r, "(\mathrm{id},\psi^p)"'] & R \times R \end{tikzcd}$$

That is, a Ψ-ring map $\alpha_p\colon R \to V_p(R)$ such that $\pi\alpha_p = (\mathrm{id}, \psi^p)$ corresponds exactly to a θ^p-ring structure on (R, ψ^p) such that $\psi^q\theta^p = \theta^p\psi^q$ for all primes $q \neq p$, by the discussion in §3.1. (Recall that for any θ^p-ring structure on (R, ψ^p) we automatically have that $\psi^p\theta^p = \theta^p\psi^p$.) A choice of such α_p for all primes p thus amounts to a λ-ring structure on R by (4.10).

Clearly, a necessary condition to lift Ψ-ring structure on R to a λ-ring structure is that the Ψ-ring structure should satisfy the Frobenius condition. As a corollary of (3.2)(2) we obtain Wilkerson's criterion: if R is torsion free as an abelian group, then any Ψ-ring structure on R satisfying the Frobenius condition lifts uniquely to a λ-ring structure.

5.3. **The relative lifting problem for λ-rings.** Now consider the following problem. Suppose we are given a λ-ring A , a Ψ-ring B which satisfies the Frobenius condition, and a homomorphism $f\colon A \to B$ of Ψ-rings. What are the possible λ-ring structures on B making f a map of λ-rings?

5.4. **Proposition.** *If A is a λ-ring, B a Ψ-ring satisfying the Frobenius condition, and $f\colon A \to B$ is a map of Ψ-rings which is formally etale as a map of commutative rings, then there exists a unique λ-ring structure on B making f a map of λ-rings.*

Proof. In view of (3.7), there are unique θ^p-ring structures on B making f a map of θ^p-rings for each p. In view of (3.8), these θ^p-structures on B must commute with the given ψ^q-operations on B for $q \neq p$. That is, apply (3.8) for θ^p-rings to the commutative triangle of ring maps

$$\begin{tikzcd} A \arrow[r, "f\psi^q_A"] \arrow[d, "f"'] & B \\ B \arrow[ur, "\psi^q_B"'] & \end{tikzcd}$$

where f and thus $f\psi^q_A$ are θ^p-ring maps, to show that $\psi^q_B \colon B \to B$ is a map of θ^p-rings. $\square$

5.5. **Proposition.** *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B \arrow[ur, "h"'] & \end{tikzcd}$$

of commutative ring maps, such that (i) A, B, and C are λ-rings, (ii) f and g are maps of λ-rings, (iii) h is a map of Ψ-rings, and (iv) f is formally etale. Then h is a map of λ-rings.

Proof. Apply (3.8) to show that h is a map of θ^p-rings for all p. $\square$

Given a λ-ring A, let $\lambda\mathrm{Ring}(A)$ denote the category of λ-rings under A, and $\Psi\mathrm{Ring}_{\mathrm{Fr}}(A)$ the category of Ψ-rings under A which satisfy the Frobenius congruence. Let $\lambda\mathrm{Ring}(A)_{\mathrm{f.etale}}$ and $\Psi\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ denote the respective full subcategories consisting of objects $f \colon A \to B$ such that f is formally etale.

5.6. **Proposition.** *The evident forgetful functor $\lambda\mathrm{Ring}(A)_{\mathrm{f.etale}} \to \Psi\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ is an equivalence of categories.*

The statement of the previous proposition remains true if we replace "formally etale" with any subclass of maps, such as "etale" or "weakly etale".

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