BROWN-COMENETZ DUALITY AND THE ADAMS SPECTRAL SEQUENCE

MARK MAHOWALD AND CHARLES REZK

Abstract. We show that the class of p-complete connective spectra with finitely presented cohomology over the Steenrod algebra admits a duality theory related to Brown-Comenetz duality. This construction also produces a full-plane version of the classical Adams spectral sequence for such spectra, which converges to the homotopy groups of a "finite" localization.

1. Introduction

In the paper [3], Brown and Comenetz introduced a notion of duality into stable homotopy. In [4] Hopkins and Gross showed that this notion in certain situations is closely connected with Spanier-Whitehead duality. In this note we wish to explore this connection and investigate it in connection with Adams spectral sequence considerations. In particular, we study a class of spectra which we call *fp-spectra* (Section 3). These are connective, p-complete spectra whose mod p cohomology is *finitely presented* over the Steenrod algebra; that is, the cohomology of such a spectrum is described by a finite set of generators together with a finite set of relations. This class of spectra includes the Johnson-Wilson spectra $BP\langle n\rangle$, connective K-theories, and the "higher" connective K-theory spectrum eo_2. The class of fp-spectra also includes some objects whose Bousfield L_n-localizations are the same as L_n localizations of finite complexes, at least in some cases. A classical example is the connective image-of-J spectrum, whose L_1-localization is L_1S^0. It follows from calculations of Shimomura and Yabe that a -1-connective cover of L_2S^0 at primes $p \geq 5$ is also an fp-spectrum (Proposition 3.7).

We show that the category of fp-spectra admits a notion of duality (Theorem 8.11). This duality is related to both Brown-Comenetz duality and Spanier-Whitehead duality. The dual WX of an fp-spectrum X will be defined to be the Brown-Comenetz dual of the fiber of the map $X \to L_n^fX$ to the "finite localization" of X (for n sufficiently large). The dual WX is itself an fp-spectrum. This duality is related to Spanier-Whitehead duality through its action on cohomology, in the following sense. If $H^*X \approx A^* \otimes_{A^*(n)} M$ where $A^*(n) \subset A^*$ is a finite sub-Hopf algebra of the Steenrod algebra, and M is a finite $A^*(n)$ module, then $H^*WX \approx A^* \otimes_{A^*(n)} \check{M}$, where $\check{M} \approx \hom_{\mathbb{F}_p}(M, \mathbb{F}_p)$ is the "Spanier-Whitehead dual" of M as a finite module over $A^*(n)$.

Because of this duality, the L_n^f-localization of an fp-spectrum is quite computable. We show that there is a full-plane spectral sequence computing $\pi_*L_n^fX$, with E_2-term a "Tate cohomology" of H^*X as a module over the Steenrod algebra

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(Proposition 6.3 and Theorem 7.1). In all cases we know of $L_n^f X \approx L_n X$ for an fp-spectrum X.

1.1. **Organization of the paper.** In Section 2 we discuss modules and comodules which are finitely presented over the Steenrod algebra. In Section 3 we define the notion of fp-spectra, and give examples. In Section 4 we discuss a duality functor for finitely presented comodules over the Steenrod algebra, which is related to the action of Brown-Comenetz duality on Eilenberg-Mac Lane spectra discussed in Section 5. In Section 6 we note that an fp-spectrum admits a tower associated to a spectral sequence whose E_2-term is the "Tate cohomology" of the homology of the spectrum, and in Section 7 show that such a tower realizes the localization functor L_n^f. In Section 8 we describe the duality theory of fp-spectra. In Section 9 we calculate some examples.

1.2. **Notation.** In this paper we work at one prime p at a time. We let A^* denote the mod p Steenrod algebra, and A_* denote the dual mod p Steenrod algebra.

Unless otherwise indicated, all vector spaces, modules, and comodules in this paper are graded. If V is a graded vector space over $\mathbb{F}_p$, then $\check{V}$ denotes the linear dual $\hom(V, \mathbb{F}_p)$. If V is a left comodule over a graded Hopf algebra B, then $\check{V}$ is taken to be a left comodule over B, via the canonical anti-automorphism χ of B.

When dealing with graded objects, we use the following sign convention: a sign is introduced whenever two symbols of odd degree are commuted.

2. Finitely presented modules and comodules over the Steenrod algebra

A module M over the mod p Steenrod algebra A^* is called **finitely presented** if it fits in an exact sequence of modules

$$A^* \otimes V_1 \to A^* \otimes V_0 \to M \to 0$$

where V_i for $i = 0, 1$ are finite dimensional graded $\mathbb{F}_p$-vector spaces. Likewise, a comodule N over the dual mod p Steenrod algebra A_* is called **finitely presented** if it fits in an exact sequence of comodules

$$0 \to N \to A_* \otimes V_0 \to A_* \otimes V_1$$

where V_i for $i = 0, 1$ are finite dimensional graded $\mathbb{F}_p$-vector spaces. Because all finitely presented modules and comodules are of finite type, we can pass easily between comodule and module language by taking vector space duals.

The Steenrod algebra A^* is a union of finite-dimensional sub-Hopf algebras. For example, $A^* = \bigcup_n A^*(n)$, where $A^*(n) \subset A^*$ is a finite dimensional sub-Hopf algebra of the Steenrod algebra, generated as an algebra by $\{ Sq^{2^i} \mid i \leq n+1 \}$ if $p = 2$ and by $\{ \beta, P^{p^i} \mid i \leq n \}$ if p is odd. Recall that A^* is free as an $A^*(n)$-module.

Lemma 2.1.

1. *A module M over A^* is finitely presented if and only if it is of the form $M \approx A^* \otimes_E N$ for some finite dimensional sub-Hopf algebra $E \subset A^*$ and some finite dimensional E-module N.*
2. *Every map $f\colon M \to M'$ of finitely presented A-modules is of the form $f \approx A^* \otimes_E g\colon A^* \otimes_E N \to A^* \otimes_E N'$ for some finite dimensional sub-Hopf algebra $E \subset A^*$ and some map $g\colon N \to N'$ of finite dimensional E-modules.*

Proof. Any finite sub-Hopf algebra $E \subset A^*$ is contained in $A^*(n)$ for some $n \geq 0$, whence part 1 is [12, Ch. 13, Prop. 2(a)]. Part 2 follows by similar arguments. □

Proposition 2.2.

1. *The kernel and cokernel of a map of finitely presented A^*-modules are finitely presented.*
2. *A retract of a finitely presented A^*-module is finitely presented.*
3. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in which M' and M'' are finitely presented, then M is also finitely presented.*

Proof. Since A^* is $A^*(n)$-free, the functor $A^* \otimes_{A^*(n)} (-)$ is exact, and hence part 1 follows from (2.1).

To prove part 2, note that a retract N of M is the kernel of an idempotent self-map $e\colon M \to M$. Hence part 2 follows from part 1.

The proof of part 3 is a standard result about finitely presented modules over any ring. □

Proposition 2.3. *Suppose M is an A^*-module and F is a* finite *A^*-module. Then M is finitely presented if and only if $M \otimes F$ is.*

Proof. It is immediate from (2.2) that $M \otimes F$ is finitely presented if M is, since F admits a finite filtration whose subquotients are copies of $\mathbb{F}_p$.

Suppose $M \otimes F$ is finitely presented. Since F is a finite module, we can choose a "pinch" map $\pi\colon F \to \Sigma^d \mathbb{F}_p$ to a "bottom-dimensional cell" of F, and we can write $i\colon \bar{F} \to F$ for the kernel of π. Then there is an exact sequence

$$M \otimes \bar{F} \otimes F \xrightarrow{1 \otimes i \otimes \pi} M \otimes F \xrightarrow{1 \otimes \pi} M \to 0$$

which exhibits M as a cokernel of a map between finitely presented modules, and the result follows from (2.2). □

Remark 2.4. Note that (2.1), (2.2), and (2.3) dualize to similar statements about finitely presented comodules. We will not state the dual form of these results, although we will make use of them in what follows.

2.5. **Homological algebra for finitely presented comodules.** Henceforth we concentrate on finitely presented comodules. We let $\mathcal{M}_{\mathrm{fp}}$ denote the category of finitely presented comodules over A_*. By (2.2) we see that $\mathcal{M}_{\mathrm{fp}}$ is an abelian category.

Proposition 2.6. *The dual Steenrod algebra A_*, viewed as a A_*-comodule, is both projective and injective in $\mathcal{M}_{\mathrm{fp}}$, and $\mathcal{M}_{\mathrm{fp}}$ has enough projectives and injectives.*

Proof. It is clear that A_* is injective in the full category of A_* comodules, and hence A_* is injective in $\mathcal{M}_{\mathrm{fp}}$ and there are enough injectives in $\mathcal{M}_{\mathrm{fp}}$. To prove that A_* is projective, consider a surjection $M \to M'$ of finitely presented comodules. By (2.1), this map is extended up from a surjection $N \to N'$ of finite $A_*(n)$-comodules. Since $\hom_{A_*}(A_*, M) \approx \hom_{A_*(n)}(A_*, N)$ and A_* is $A_*(n)$-free, any map $A_* \to M'$ can be lifted to a map to M. Furthermore, we can always produce enough maps from A_* to a finitely presented comodule, and thus $\mathcal{M}_{\mathrm{fp}}$ has enough projectives. □

Remark 2.7. It is known that the Steenrod algebra is injective as an A^*-module over itself [12, p. 201]. It would be interesting to know whether A_* is projective as a comodule over itself, without the restriction to the finitely presented category.

Given a finitely presented comodule M, one can define its **Tate cohomology** as follows. By (2.6) we can choose injective and projective resolutions

$$0 \to M \to C_0 \to C_1 \to C_2 \to \dots$$

and

$$\dots \to C_{-3} \to C_{-2} \to C_{-1} \to M \to 0$$

by finitely generated free A_*-comodules. By gluing the ends together, we obtain an unbounded complex

$$\dots \to C_{-2} \to C_{-1} \to C_0 \to C_1 \to \dots,$$

which in each degree is injective in the category of comodules. For $s \in \mathbb{Z}$ define

$$H^s_{\mathrm{Tate}}(M) = H^s\left[\hom_{A_*}(\mathbb{F}_p, C_\bullet)\right].$$

Of course, if we can write $M \approx A_* \otimes_{A_*(n)} N$ for some $A_*(n)$-comodule N, and we choose resolutions $0 \to N \to D_\bullet$ and $D_{-\bullet} \to N \to 0$ of N by finite free $A_*(n)$-comodules, then we see that

$$H^s_{\mathrm{Tate}}(M) \approx H^s\left[\hom_{A_*(n)}(\mathbb{F}_p, D_\bullet)\right].$$

These groups are the same as morphisms in the stable category of $A_*(n)$-modules, as is shown in [7, Sec. 9.6].

3. fp-spectra

In this section we define the notion of fp-spectra, and produce several examples. Recall that we work in the category of p-local spectra.

We first note the following theorem of Mitchell.

Theorem 3.1 (Mitchell)**.** [13] *For each n there exists a non-trivial finite complex F such that H_*F is $A_*(n)$-free.*

If X is a spectrum, say that π_*X is **finite** if $\pi_kX = 0$ for all but finitely many $k \in \mathbb{Z}$, and is a finite group otherwise.

Proposition 3.2. *Suppose X is a connective, p-complete spectrum. Then the following are equivalent.*

1. *H_*X is finitely presented as a comodule over the Steenrod algebra.*
2. *$H_*X \approx A_* \otimes_{A_*(n)} M$ for some $n \geq 0$ and some finite $A_*(n)$-comodule M.*
3. *There exists a non-trivial finite complex F such that $X \wedge F$ is a finite wedge of suspensions of mod p Eilenberg-Mac Lane spectra.*
4. *There exists a non-trivial finite complex F such that $\pi_*(X \wedge F)$ is finite.*

Proof. The equivalence of 1 and 2 is just (2.1). Likewise, 4 is immediate given 3.

To show that 2 implies 3, we let F be as in (3.1), with H_*F free over $A_*(n)$. Thus $H_*(X \wedge F)$ is free over the Steenrod algebra on a finite set of generators, whence $X \wedge F$ is a wedge of mod-p Eilenberg-Mac Lane spectra $H\mathbb{F}_p$.

To show that 4 implies 1, note that if π_*Y is finite for a spectrum Y, then Y can be built from finitely many copies of $H\mathbb{F}_p$, whence H_*Y is finitely presented by (2.2). Thus if $\pi_*(X \wedge F)$ is finite, then $H_*(X \wedge F) \approx H_*X \otimes H_*F$ is finitely presented, and hence H_*X is finitely presented by (2.3). ∎

We call a spectrum X an **fp-spectrum** if it is connective, p-complete, and satisfies any of the four equivalent statements of (3.2). Let $\mathcal{C}$ denote the class of all fp-spectra. This class includes the Eilenberg-Mac Lane spectra $H\mathbb{Z}/p^n$ and $H\mathbb{Z}_p$, the p-completed Johnson-Wilson spectrum $BP\langle n\rangle$, which has $\pi_*BP\langle n\rangle \approx \mathbb{Z}_p[v_1,\dots,v_n]$, and connective Morava K-theories. Non-trivial suspension spectra, and in particular finite complexes, are not fp-spectra.

Recall that a finite complex F is of **type** n if $K(0)_*F \approx \cdots \approx K(n-1)_*F \approx 0$ and $K(n)_*F \not\approx 0$, where $K(m)$ denotes the mth Morava K-theory. Define the **fp-type** of an fp-spectrum X by

$$\operatorname{fptype}(X) = \min\{(\operatorname{type}(F)-1) \text{ such that } \pi_*(X\wedge F) \text{ is finite}\}.$$

By the thick subcategory theorem [6], if $\operatorname{fptype}(X) = n$ then $\pi_*(X\wedge F)$ is finite for all F of type $> n$. Thus, $\operatorname{fptype}(H\mathbb{F}_p) = -1$, $\operatorname{fptype}(H\mathbb{Z}_p) = 0$, and $\operatorname{fptype}(BP\langle n\rangle) = n$. Furthermore, if $H_*X \approx A_* \otimes_{A_*(n)} M$, then $\operatorname{fptype}(X) \le n$.

Let $\mathcal{C}_n$ denote the class of fp-spectra of type $\le n$. Then $\mathcal{C}_n$ is a subcategory of the category of spectra, and $\mathcal{C}_n \subset \mathcal{C}_{n+1}$.

Proposition 3.3. *The classes $\mathcal{C}$ and $\mathcal{C}_n$ for $n \ge -1$ are thick subcategories of the homotopy category of spectra.*

Proof. That $\mathcal{C}$ is a thick subcategory is an immediate consequence of criterion 4 of (3.2). Alternately, this follows from criterion 1 of (3.2) together with (2.2). The proof for $\mathcal{C}_n$ is similar. □

For example, $\mathcal{C}_{-1}$ is the class of all p-complete spectra X with π_*X finite; as a thick subcategory it is generated by $H\mathbb{F}_p$. Likewise, $\mathcal{C}_0$ is the class of all fp-spectra which are finite Postnikov towers; as a thick subcategory it is generated by $H\mathbb{Z}_p$. The class $\mathcal{C}_1$ contains the p-completed connective K-theory spectra bo and bu, along with their connective covers. Thus $\mathcal{C}_1$ contains the image of J spectrum, since $J \approx \operatorname{fib}(bo \to bspin)$. The class $\mathcal{C}_2$ contains eo_2, the connective version of the "higher real K-theory" spectrum EO_2 of Hopkins and Miller.

3.4. **L_n-localization and fp-spectra.** Let L_n denote Bousfield localization with respect to the wedge $K(0)\vee\cdots\vee K(n)$ of Morava K-theories. A spectrum W is **L_n-local** if $L_nW \approx W$.

Proposition 3.5. *Let W be an L_n-local spectrum such that for each $k \in \mathbb{Z}$ the homotopy group π_kW has the form*

$$\pi_kW \approx F_k \oplus \mathbb{Z}_p^{\oplus a_k} \oplus (\mathbb{Q}/\mathbb{Z}_{(p)})^{\oplus b_k} \oplus \mathbb{Q}_p^{\oplus c_k},$$

where F_k is a finite p-group, $a_k = 0 = c_k$ for all sufficiently small $k \ll 0$, and $b_k = 0 = c_k$ for all sufficiently large $k \gg 0$.

Then there exists a map $f\colon X \to W$ such that X is an fp-spectrum of fp-type n and $L_nX \to L_nW \approx W$ is a weak equivalence.

Proof. Consider the connected cover $Y = W\langle -N,\dots,\infty\rangle$, where N is chosen so that $a_k = c_k = 0$ for $k < -N$. If F is a finite complex with bottom cell in dimension 0 and top cell in dimension d, then the map $Y\wedge F \to W\wedge F$ is an isomorphism on π_k for $k > d-N$, as can be seen by comparing the Atiyah-Hirzebruch spectral sequences computing Y_*F and W_*F.

If F is a type $(n+1)$ complex, then $W\wedge F \approx *$, and so $Y\wedge F$ has non-trivial homotopy in only a finite range of dimensions $(-N,\dots,d-N)$, and each homotopy group is finite. Thus we have found a connective spectrum Y which satisfies criterion 4 of (3.2), and furthermore $L_nY \approx W$, since the fiber of $Y \to W$ is coconnective with torsion homotopy and thus is killed by L_n.

In order to get a p-complete spectrum, it suffices to replace any copies of $\mathbb{Q}/\mathbb{Z}_{(p)}$ or $\mathbb{Q}_p$ in the homotopy of Y by a finite torsion group or a copy of $\mathbb{Z}_p$ respectively; by hypothesis there are only finitely many such copies to worry about. Note that $[\Sigma^{-i}HA, H\mathbb{Q}/\mathbb{Z}_{(p)}]$ is a finite torsion group if A is a finitely generated $\mathbb{Z}_p$-module and $i > 0$; thus by induction on the Postnikov tower of Y we can see that we can always find a map

$$Y \to \bigvee_{i=1}^{r} \Sigma^{n_i} H\mathbb{Q}/\mathbb{Z}_{(p)}$$

which is surjective on homotopy, and so that the fiber X of this map is p-complete. Then since $L_nH\mathbb{Q}/\mathbb{Z}_{(p)} \approx *$ we see that X is the desired spectrum. □

Remark 3.6. In view of Conjecture (7.3) and of (8.9), it seems likely that the converse of (3.5) should hold. That is, if X is an fp-spectrum with $\text{fptype}(X) = n$, then we expect that π_kL_nX has the form given in (3.5).

It is interesting to know when the L_n-localization of a finite complex F is also the L_n-localization of an fp-spectrum X. One can say the following.

Proposition 3.7. *If F is the p-completion of a finite complex, then L_nF is the L_n-localization of an fp-spectrum X in the following cases,*

1. *$n = 0$ or $n = 1$ (at any prime),*
2. *$n = 2$ if $p \geq 5$, or*
3. *for any n and any prime p if F is a type n complex.*

Proof. It suffices to show in each case that the hypotheses of (3.5) are satisfied.

The cases $n = 0$ and $n = 1$ are well known. In fact, for $n = 0$ take $X = H\mathbb{Z}_p \wedge F$, and for $n = 1$ take $X = J \wedge F$, where J is the connective image-of-J spectrum.

If n is arbitrary, but F is a type n complex, then the hypotheses of (3.5) hold. This is a consequence of the fact that the cohomology of the Morava stabilizer algebra is a finitely generated algebra (see [15, Thm. 6.2.10]), together with Hopkins and Ravenel's demonstration of a horizontal vanishing line at the E_∞-term of the Adams-Novikov spectral sequence of L_nF (see [16, Section 8.3]). These imply that π_kF is finite for all k.

When $n = 2$ and $p \geq 5$, one can take a spectrum of the form $Y \wedge F$, where Y is an fp-spectrum such that $L_2Y \approx L_2S^0$. We show that the hypotheses of (3.5) hold for L_2S^0 at $p \geq 5$.

First, we note that the hypotheses of (3.5) hold for $L_2M(p)$, the localization of the mod p Moore space. This is a consequence of calculations of Shimomura [18], as we explain below. There is a diagram

$$\begin{array}{ccccc} L_2M(p) & \longrightarrow & L_1M(p) & \longrightarrow & L_2M(p, v_1^\infty) \\ \downarrow & & \downarrow & & \downarrow \sim \\ L_{K(2)}M(p) & \longrightarrow & L_1L_{K(2)M(p)} & \longrightarrow & L_2M(p, v_1^\infty) \end{array}$$

in which the left-hand square is a pull-back square; this is because all the objects in it are L_2-local, and the square is a pull-back after smashing with $K(0) \vee K(1) \vee K(2)$.

Since $L_1M(p) \approx L_1(v_1^{-1}M(p)) \approx L_2(v_1^{-1}M(p))$, the top row is a cofiber sequence, and thus so is the bottom row.

Shimomura computes the E_2-term of the Adams-Novikov spectral sequence for $L_2M(p, v_1^\infty)$. Using Shimomura's calculation one may (with careful analysis) read off that each group $\pi_k L_2M(p, v_1^\infty)$ must be finite; see the presentation of the results of this calculation given in [17]. Since $\pi_k L_1 M(p)$ is known to be finite, this shows that $\pi_k L_2 M(p)$ must be finite.

Hovey and Strickland [8, Thm. 15.1] actually carry out the careful analysis to show that each group $\pi_k L_{K(2)}M(p)$, $k \in \mathbb{Z}$, is finite, so we will derive what we need from their results. To derive the finiteness of $\pi_k L_2M(p)$, it suffices to show that $L_1L_{K(2)}M(p)$ has finite homotopy groups. This spectrum is equivalent to $v_1^{-1}L_{K(2)}M(p)$, since the v_1-self map of $M(p)$ is trivial on $K(2)_*M(p)$. It happens that $\pi_*L_{K(2)}M(p)$ decomposes as a finite sum of copies $\mathbb{F}_p[v_1]$ plus a summand which is v_1-torsion. Thus $\pi_*v_1^{-1}L_{K(2)}M(p)$ is a finite sum of copies of the form $\mathbb{F}_p[v_1^{\pm}]$, which is clearly a finite group in each dimension.

When $n = 2$ and $p \geq 5$, the hypotheses of (3.5) hold for L_2S^0; this is a consequence of the above remarks together with the calculation of Shimomura and Yabe [19] of $\pi_*L_2S^0$ at $p \geq 5$. They show that the homotopy of L_2S^0 consists of a free summand in dimension 0, summands of the form $\mathbb{Q}/\mathbb{Z}_{(p)}$ in stems -3, -4, and -5, together with a summand T consisting of non-infinitely divisible torsion. The above remarks on the finiteness of $\pi_kL_2M(p)$ imply that the summand T of $\pi_*L_2S^0$ is finite in each stem. □

We are led to make the following conjectures.

Conjecture 3.8. *The hypotheses of (3.5) hold for any L_n-localization of a finite complex, for any $n \geq 0$.*

Conjecture 3.9. *For every finite complex F there exists an fp-spectrum X such that $L_nF \approx L_nX$.*

Of course, Conjecture (3.8) implies Conjecture (3.9) as we have shown above. There is also reason to believe that Conjecture (3.9) would imply Conjecture (3.8); see section (7.2).

3.10. **Adams towers for fp-spectra.** For a spectrum X we can construct an Adams tower. This is a tower of spectra $\ldots \to X_{s+1} \to X_s \to \ldots$ with $X_0 = X$ and with fiber sequences $X_{s+1} \to X_s \xrightarrow{k_s} \Sigma^{-s}HV_s$, where k_s is injective on mod p homology; hence there is a resolution

$$0 \to H_*X \to H_*HV_\bullet,$$

and an Adams spectral sequence with $E_2^{s,t} = \operatorname{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*X)$.

If X is an fp-spectrum, we can choose an Adams tower in which each V_s is a *finite dimensional* graded vector space. We call such an **fp-Adams resolution**. In fact, if $H_*X \approx A \otimes_{A_*(n)} M$ for some finite $A_*(n)$-module M, then the chain complex $H_*HV_\bullet$ is induced from a resolution of M by $A_*(n)$-comodules.

Given an Adams tower $\{X_s\}$, we may produce another tower

$$\ldots \to X_0^s \to X_0^{s-1} \to \ldots \to X_0^1 \to X_0^0 \to *$$

by taking cofibers $X_{s+1} \to X \to X_0^s$. The fibers in this tower are $\Sigma^{-s}HV_s \to X_0^s \to X_0^{s-1}$. Note that in the E_2-term of the spectral sequence for this tower, each horizontal line $E_2^{s,*}$ is finite dimensional, and thus the spectral sequence satisfies the *complete convergence* condition of [2, p. 263]. Thus the homotopy inverse limit $\mathrm{holim}_s X_0^s$ does not depend on the choice of fp-Adams resolution, and by a result of Bousfield [1, Prop. 5.8 and Thm. 6.6] is equivalent to X, since X is p-complete and connective.

4. Duality for finitely presented comodules

In this section we describe a duality functor $\tilde{I}$ on the category of finitely presented comodules. This functor was essentially introduced by Brown and Comenetz in [3]. Our interest in this functor stems from the fact that it generalizes the notion of "Spanier-Whitehead duality" of finite comodules over $A_*(n)$, in which a comodule is dual to its vector space dual. It will be needed in later sections to study duality on fp-spectra.

4.1. **Construction of functors $\tilde{J}$ and $\tilde{I}$.** To motivate the construction, note that if X is a spectrum and H denotes the mod p Eilenberg-Mac Lane spectrum, then the graded vector space $[H, X]_*$ is in a natural way a module over the Steenrod algebra. This object is approximated by the edge map $[H, X]_* \to \hom_{A_*}(H_*H, H_*X)$ of the Adams spectral sequence. This algebraic approximation itself admits an action by the Steenrod action, and we will call this module $\tilde{J}(H_*X)$. The vector space dual to $\tilde{J}(H_*X)$ will be $\tilde{I}(H_*X)$, which admits a coaction by the dual Steenrod algebra.

Recall that if we regard the dual Steenrod algebra A_* as a left-comodule over itself, then each element $a \in A^*$ of the Steenrod algebra induces a map $A_* \to A_*$ of left A_*-comodules via

$$a \cdot z = \sum (-1)^{|a||z|} z' \langle z'', a \rangle,$$

where $z \in A_*$, $\sum z' \otimes z''$ is the diagonal of z in $A_* \otimes A_*$, and $\langle z, a \rangle$ represents the usual pairing of A_* and A^*. In fact, there is an isomorphism of algebras

$$\hom_{A_*}(A_*, A_*) \approx A^*. \tag{4.2}$$

We reserve the notation $a \cdot z$ for this action. Note that this action gives A_* the structure of a left A^*-module; however, this is not identical to the usual left action of the Steenrod algebra on H_*H.

We define a functor $\tilde{J}$ from left A_*-comodules to left A^*-modules by

$$\tilde{J}(M) = \hom_{A_*}(A_*, M).$$

This has a natural right A^*-action induced by pre-composition of comodule maps, using (4.2), which is made into a left A^*-action using χ; if $z \in A_*$, $a \in A^*$, and $f \in \tilde{J}(M)$, the left action can be written

$$(a \cdot f)(z) = (-1)^{|a||f|} f(\chi a \cdot z).$$

Note that if M is finitely generated then $\tilde{J}(M)$ is bounded below (where we use cohomological grading for $\tilde{J}(M)$).

We define a functor $\tilde{I}$ from finitely generated left comodules to left comodules by

$$\tilde{I}(M) = (\tilde{J}(M))\check{} \approx \hom_{A_*}(A_*, M)\check{}.$$

Since $\tilde{J}(M)$ was bounded below, $\tilde{I}(M)$ is bounded below, and thus receives a left comodule structure in the usual way. This structure is characterized as follows: if

$u \in \tilde{I}(M)$ and $u \mapsto \sum u' \otimes u'' \in A_* \otimes \tilde{I}(M)$ is the comodule action on u, and if $a \in A^*$ and $f \in \tilde{J}(M)$, then

$$\langle u, a \cdot f\rangle = \sum (-1)^{|a||u''|} \langle u', a\rangle \langle u'', f\rangle.$$

In comparison, Brown and Comenetz [3] define a functor c_p on the category of A^*-modules; their functor is defined by

$$c_p(M) \approx \hom_{A^*}(M, A^*),$$

with an appropriate A^*-action. Thus, our $\tilde{I}$ is just a comodule version of their c_p.

4.3. **Action of $\tilde{J}$ and $\tilde{I}$ on free comodules.** Let $\mathcal{M}_{\text{free}} \subset \mathcal{M}_{\text{fp}}$ denote the full subcategory of comodules of the form $A_* \otimes V$, where V is a finite vector space. We want compute the action of $\tilde{J}$ and $\tilde{I}$ on this subcategory.

The following describes the category $\mathcal{M}_{\text{free}}$.

Proposition 4.4. *For finite V and W, $\hom_{A_*}(A_* \otimes V, A_* \otimes W) \approx A^* \otimes \hom(V, W)$, where $a \otimes \sigma \in A^* \otimes \hom(V, W)$ corresponds to the morphism of comodules*

$$z \otimes v \mapsto (-1)^{|\sigma||z|} a \cdot z \otimes \sigma(v), \qquad z \in A_*, a \in A^*, v \in V, \sigma \in \hom(V, W),$$

and composition of maps is given by $(a \otimes \sigma) \circ (b \otimes \tau) = (-1)^{|\sigma||b|} ab \otimes \sigma\tau$.

Proof. Straightforward. □

Proposition 4.5. *There is a natural isomorphism*

$$\tilde{J}(A_* \otimes V) \approx \hom_{A_*}(A_*, A_* \otimes V) \approx A^* \otimes V$$

of left A^*-modules, where $a \otimes v \in A^* \otimes V$ corresponds to the map defined by

$$z \mapsto (-1)^{|z||v|} \chi a \cdot z \otimes v, \qquad z \in A_*, a \in A^*, v \in V.$$

Given a map $a \otimes \sigma \colon A_* \otimes V \to A_* \otimes W$ of comodules, the induced map $\tilde{J}(a \otimes \sigma) \colon A^* \otimes V \to A^* \otimes W$ sends $b \otimes v \mapsto (-1)^{|b|(|a|+|\sigma|)} b\, \chi a \otimes \sigma(v)$.

Proof. Straightforward. □

Corollary 4.6. *There is a natural isomorphism*

$$\tilde{I}(A_* \otimes V) \approx (A^* \otimes V)\check{} \approx A_* \otimes \check{V}$$

of left-comodules. The induced map $\tilde{I}(a \otimes \sigma) \colon A_* \otimes \check{W} \to A_* \otimes \check{V}$ sends

$$z \otimes \check{w} \mapsto (-1)^{|\sigma||z|} \chi a \cdot z \otimes \check{\sigma}(\check{w}), \qquad z \in A_*, a \in A^*, \check{w} \in \check{W},$$

where $\check{\sigma} \in \hom(\check{W}, \check{V})$ is the adjoint to $\sigma \in \hom(V, W)$. In other words, $\tilde{I}(a \otimes \sigma) = \chi a \otimes \check{\sigma}$.

Remark 4.7. In terms of bases v_j and w_i for V and W we can view maps $A_* \otimes V \to A_* \otimes W$ as matrices (a_{ij}) with entries in the Steenrod algebra, acting by

$$\sum_j z_j \otimes v_j \mapsto \sum_{i,j} a_{ij} \cdot z_j \otimes w_i.$$

Hence, the induced map $\tilde{I}(a_{ij}) \colon A_* \otimes \check{W} \to A_* \otimes \check{V}$ corresponds to the matrix (χa_{ji}) in terms of the dual bases $\check{v}_j$ and $\check{w}_i$ of $\check{V}$ and $\check{W}$.

4.8. **Properties of the duality functor.**

Proposition 4.9. *The functor* $\tilde{I}$ *restricts to a functor* $\tilde{I}\colon \mathcal{M}_{\mathrm{fp}}^{\mathrm{op}} \to \mathcal{M}_{\mathrm{fp}}$*, and is exact on* $\mathcal{M}_{\mathrm{fp}}$*. Furthermore, there is a natural isomorphism* $M \to \tilde{I}\tilde{I}M$ *for objects in* $\mathcal{M}_{\mathrm{fp}}$*.*

Proof. That $\tilde{I}$ is exact on the subcategory of finitely presented comodules follows from (2.6). By (4.6) the functor $\tilde{I}$ takes finitely generated free comodules to the same, and thus by exactness $\tilde{I}(\mathcal{M}_{\mathrm{fp}}) \subset \mathcal{M}_{\mathrm{fp}}$.

One can construct a natural isomorphism $M \to \tilde{I}\tilde{I}M$ when $M \approx A_* \otimes V$, since by (4.6) $\tilde{I}\tilde{I}(A_* \otimes V)$ is tautologically isomorphic to $A_* \otimes V$. This isomorphism extends by exactness to all finitely presented comodules. □

Proposition 4.10. *Let* M *be a finite* $A_*(n)$*-comodule. Then*

$$\tilde{I}(A_* \otimes_{A_*(n)} M) \approx A_* \otimes_{A_*(n)} \Sigma^{a(n)} \check{M}$$

as left comodules, where $a(n)$ *is the dimension of the "top cell" of* $A_*(n)$*.*

Proof. There is an exact sequence

$$0 \to M \to A_*(n) \otimes V \xrightarrow{(a_{ij})} A_*(n) \otimes W$$

of $A_*(n)$-comodules, where $a_{ij} \in A^*(n)$. After applying vector space duals we can identify the resulting sequence with

$$0 \leftarrow \Sigma^{-a(n)} \check{M} \leftarrow A_*(n) \otimes \check{V} \xleftarrow{(\chi a_{ji})} A_*(n) \otimes \check{W}$$

by "Poincaré duality" of $A_*(n)$ [12, Ch. 12.2]. The result follows from (4.7) after extending up to A_*. □

5. Brown-Comenetz duality

Recall that the functor

$$X \mapsto \hom(\pi_0 X, \mathbb{Q}/\mathbb{Z})$$

is a generalized cohomology theory satisfying the wedge axiom, and hence is represented by a spectrum I. We write $IY = \mathcal{F}(Y, I)$ for the function spectrum, whence IY is the spectrum representing the functor

$$X \mapsto IY^0(X) = \hom(Y_0 X, \mathbb{Q}/\mathbb{Z}).$$

The spectrum IY is called the **Brown-Comenetz** dual of Y.

We write $DX \approx \mathcal{F}(X, S^0)$ for the Spanier-Whitehead dual of X. Note that if X is any spectrum and F is a finite complex, then the natural map $IX \wedge DF \to I(X \wedge F)$ is an equivalence.

There is a natural double-dual map $X \to IIX$. If Y is a spectrum such that each homotopy group $\pi_k Y$ is finite, then $Y \approx IIY$ via this map. Thus, given such a Y and given any spectrum X, there is a natural isomorphism

$$[X, Y] \approx [IY, IX].$$

5.1. **Eilenberg-Mac Lane spectra.** Let $H \approx H\mathbb{F}_p$ denote the mod p Eilenberg-Mac Lane spectrum. Then $IH \approx H$. In fact, by the universal coefficient theorem,

$$IH^*X \approx \hom(H_*X, \mathbb{Q}/\mathbb{Z}) \approx \hom(H_*X, \mathbb{F}_p) \approx H^*X.$$

Lemma 5.2. *Under the above identification the map*

$$I\colon [H,H]_* \to [IH,IH]_* \approx [H,H]_*$$

sends $a \in A^*$ to χa.

Proof. This is [3, Thm. 1.9(d)]. □

More generally, let V denote a finite dimensional graded $\mathbb{F}_p$-vector space, and let $\check{V} = \hom(V, \mathbb{F}_p)$ denote its vector space dual. Let HV denote the generalized Eilenberg-Mac Lane spectrum with $\pi_*HV = V$, whence $HV_*X \approx H_*X \otimes V$.

Proposition 5.3. *There is an equivalence $IHV \approx H\check{V}$, and we have isomorphisms*

$$H^*(IHV) \approx \tilde{J}(H_*HV)$$

of A^*-modules and

$$H_*(IHV) \approx \tilde{I}(H_*HV)$$

of A_*-comodules which are natural in HV.

Proof. This is immediate from (4.6) and (5.2); alternatively, it follows from [3, Thm. 1.3]. □

5.4. **Algebraic approximation.** We note that the functor $\tilde{I}$ of Section 4 serves as an algebraic "approximation" to H_*IX, at least when H_*X has finitely presented homology.

Proposition 5.5. *For each spectrum X with finitely presented homology, there is a map*

$$\iota_X\colon \tilde{I}(H_*X) \to H_*IX$$

which is natural in X. Furthermore, this map is an isomorphism when π_*X is finite.

Compare with [3, Thm. 1.13].

Proof. Given X we can choose an Adams resolution

$$X \to HV_0 \to HV_1 \to \dots,$$

in which V_0 and V_1 are finite dimensional vector spaces, and the sequence

$$0 \to H_*X \to H_*HV_0 \to H_*HV_1$$

is exact. Applying I to the first diagram gives maps

$$IX \leftarrow H\check{V}_0 \leftarrow H\check{V}_1$$

and a sequence

$$H_*IX \leftarrow \tilde{I}(H_*HV_0) \leftarrow \tilde{I}(H_*HV_1),$$

not necessarily exact. We let ι_X be the induced map

$$\iota\colon \tilde{I}(H_*X) = \operatorname{Cok}\Big(\tilde{I}(H_*HV_1) \to \tilde{I}(H_*HV_0)\Big) \to H_*IX.$$

To see that ι_X is independent of the choice of resolution and is natural, use a map between Adams resolutions.

The map ι is by construction an isomorphism for $X = H\mathbb{F}_p$. By the exactness property of (4.9), we see that ι_X is an isomorphism for any X in the thick subcategory generated by $H\mathbb{F}_p$, which are precisely the X with finite homotopy. □

6. Geometric realization of Tate cohomology

In this section we note that one can construct for each fp-spectrum a $\mathbb{Z}$-indexed Adams tower; this is a tower which extends both above and below X, whose layers are finite mod p generalized Eilenberg-Mac Lane spectra, and which leads to a spectral sequence whose E_2-term is the Tate cohomology of H_*X. We give several constructions, starting with the most general.

6.1. **Construction of the tower.**

Lemma 6.2. *Let X be an fp-spectrum. Then*

$$[H, X]_* \approx \hom_{A_*}(A_*, H_*X) \approx \tilde{J}(H_*X).$$

Proof. Choose an fp-Adams tower $\{X_s\}$ for X. Then there is a spectral sequence $E_1^{s,t} = [H, HV_s]_t \Longrightarrow [H, X]_{t-s}$. We claim that

1. $E_2^{s,t} \approx 0$ for $s > 0$, so that $E_2^{0,t} \approx E_\infty^{*,t}$, and
2. $E_2^{0,t} \approx \hom_{A_*}(A_*, H_*X)$.

Since X is connective and p-complete, the first claim implies that the spectral sequence converges, and thus $E_2^{0,t} \approx [H, X]_t$.

To prove the claim, recall that the resolution $0 \to H_*X \to H_*HV_s$ is extended up from a resolution $0 \to M \to C(s)$ of $A_*(n)$-modules. Now

$$[H, HV_s]_* \approx \hom_{A_*}(A_*, A_* \otimes_{A_*(n)} C(s)) \approx \hom_{A_*(n)}(A_*, C(s)).$$

Since as an $A_*(n)$-comodule $A_* \approx A_*(n) \otimes A_* /\!\!/ A_*(n)$ we see that A_* is projective as an $A_*(n)$-comodule, and thus the sequence

$$0 \to \hom_{A_*}(A_*, H_*X) \to \hom_{A_*}(A_*, H_*HV_\bullet)$$

is exact as desired. □

Thus, given an fp-spectrum X, we may construct a tower "realizing" the Tate cohomology by constructing fp-spectra X_{-s} for $s \geq 0$ inductively as follows. First, let $X_0 = X$. Since $H_*X_{-s} \approx A_* \otimes_{A_*(n)} M(-s)$ for some finite module $M(-s)$, we can choose a surjection $C(-s-1) \to M(-s)$ from a free $A_*(n)$-comodule. This may be extended to a surjection $A_* \otimes_{A_*(n)} C(-s-1) \to A_* \otimes_{A_*(n)} M(-s)$, which in turn by the above lemma is realized by a map of spectra $\Sigma^s HV_{-s-1} \to X_{-s}$. If we take the cofiber

$$\Sigma^s HV_{-s-1} \to X_{-s} \to X_{-s-1}$$

we get another fp-spectrum X_{-s-1}; iteration produces an infinite sequence $\ldots \to X_{-s} \to X_{-s-1} \to \ldots$. If we put this sequence together with an fp-Adams tower for X, we get a tower $\{X_s\}_{s\in\mathbb{Z}}$, which we call a **$\mathbb{Z}$-indexed Adams tower** for X.

Note that given a map $f\colon X \to Y$ of fp-spectra and given $\mathbb{Z}$-indexed Adams towers $\{X_s\}$ and $\{Y_s\}$, we can extend f to a map of towers, by the "dual" of the usual argument, using (6.2). In particular, given any two $\mathbb{Z}$-indexed Adams towers for X we can produce a map between them.

For such a tower, let $\widehat{X} = \operatorname{hocolim}_{s\to\infty} X_{-s}$. There is a full-plane Adams-type spectral sequence $E_2^{s,t} \approx H_{\mathrm{Tate}}^{s,t}(H_*X) \Longrightarrow \pi_{t-s}\widehat{X}$ approximating the homotopy of $\widehat{X}$, where $H_{\mathrm{Tate}}^{s,t}$ is the Tate cohomology of (2.5).

Given such a tower $\{X_s\}$, write $X_p^q = \operatorname{cof}(X_{q+1} \to X_p)$ for $-\infty < p \le q < \infty$. Then let $X_\infty^{-1} = \operatorname{cof}(X \to \widehat{X}) \approx \operatorname{hocolim}_{s\to\infty} X_{-s}^{-1}$. A straightforward convergence argument shows that the homotopy spectral sequence of this colimit converges, and hence X_∞^{-1} does not depend on the choice of the $\mathbb{Z}$-indexed tower.

The above remarks are summarized in the following proposition.

Proposition 6.3.

1. *For each fp-spectrum X there exists a $\mathbb{Z}$-indexed Adams tower $\{X_s\}$.*
2. *Given $\mathbb{Z}$-indexed Adams towers $\{X_s\}$ and $\{Y_s\}$ for X and Y, and a map $f\colon X \to Y$, there exists a map $\{f_s\colon X_s \to Y_s\}$ of towers which extends f.*
3. *The colimit $\widehat{X} = \operatorname{hocolim}_{s\to\infty} X_{-s}$ of a $\mathbb{Z}$-indexed Adams tower depends only on X, and not on the choice of tower.*

6.4. **Alternate construction of the tower.** Let X be an fp-spectrum, and let F be a finite complex such that $H_*(X) \otimes H_*(F)$ is a free A_*-comodule. Suppose also a map $S^0 \to F$ which is injective on homology. Such an F always exists; if H_*X is an extended $A_*(n)$-comodule, let F be a Mitchell complex of type n as in (3.1), and $S^0 \to F$ the inclusion of a bottom cell.

Form a fiber sequence $\bar{F} \to S^0 \to F$, so that by Spanier-Whitehead duality we get a dual fiber sequence $DF \to S^0 \to D\bar{F}$. We obtain a $\mathbb{Z}$-indexed Adams tower with

$$X_s = \begin{cases} X \wedge \bar{F}^{(s)} & \text{if } s \ge 0 \\ X \wedge (D\bar{F})^{(-s)} & \text{if } s < 0, \end{cases}$$

where the maps in the tower are induced by fiber sequences $X \wedge \bar{F}^{(s+1)} \to X \wedge \bar{F}^{(s)} \to X \wedge \bar{F}^{(s)} \wedge F$ and $X \wedge D\bar{F}^{(s)} \wedge DF \to X \wedge D\bar{F}^{(s)} \to X \wedge D\bar{F}^{(s+1)}$.

In those cases when there actually exists a complex F with $H_*F \approx A_*(n)$, and H_*X is an extended $A_*(n)$-module, then this complex realizes the unbounded chain complex obtained by gluing together the bar complex for M and the co-bar complex for M as an $A_*(n)$-comodule.

6.5. **An explicit construction for *bo*.** Recall that $H^*bo \approx A^*/A^*(Sq^1, Sq^2)$. Consider the minimal Adams tower $\{bo^s\}$ for bo at $p = 2$; write $H^*bo^s \approx A^* \otimes_{A^*(1)} M(s)$.

Let $R(n)$ denote the cofiber of $P^n \to S^0$ for $n \ge 0$; thus $R(0) \approx S^0$. Recall from [9] that

$$M(4s) \approx H^*R(8s).$$

We define a map $R(8s+8) \to R(8s)$ as follows. Let $p\colon R(8s+8) \to \Sigma P_{8s+1}^{8s+8}$ denote the pinch map obtained by taking the quotient of $R(8s) \to R(8s+8)$. Then $p \circ (16) \sim 0$, where (16) is the degree 16 map on $R(8s+8) \to R(8s+8)$, and so (16) lifts to a map $f_s\colon R(8s+8) \to R(8s)$. The map f_s has degree 16 on the bottom cell.

Likewise, we may consider the Spanier-Whitehead dual map

$$f_{-s} = Df_{s-1}\colon DR(8s-8) \to DR(8s).$$

This map has degree 16 on the top cell. Note that $f_{-1}\colon S^0 \to DR(8s)$, so we can put all the f_s for $s \in \mathbb{Z}$ together into a $\mathbb{Z}$-indexed tower

$$\dots \xrightarrow{f_2} R(16) \xrightarrow{f_1} R(8) \xrightarrow{f_0} R(0) \approx S^0 \xrightarrow{f_{-1}} DR(8) \xrightarrow{f_{-2}} DR(16) \xrightarrow{f_{-3}} \dots.$$

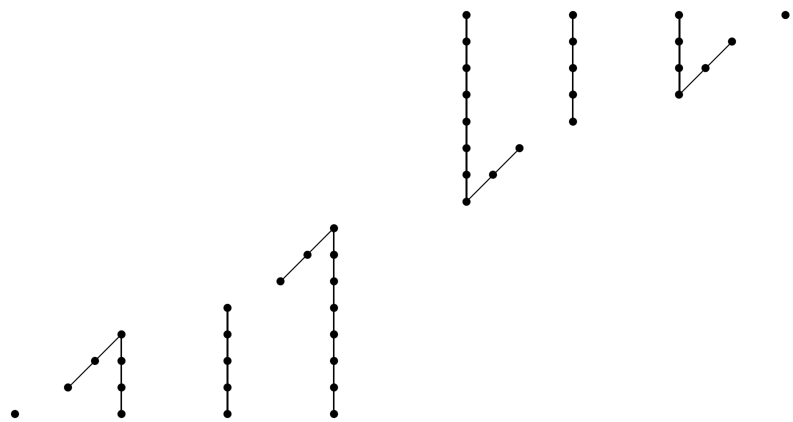
Proposition 6.6. *There is a $\mathbb{Z}$-indexed Adams tower $\{bo^s\}$ for bo with*

$$bo^{4s} \approx \begin{cases} bo \wedge R(8s) & \text{if } s \geq 0, \\ bo \wedge DR(-8s) & \text{if } s < 0, \end{cases}$$

and the map $bo^{4s+4} \to bo^{4s}$ is $bo \wedge f_s$.

Proof. Set bo^{4s} as above, and construct a the start of a minimal Adams tower $bo^{4s+3} \to bo^{4s+2} \to bo^{4s+1} \to bo^{4s}$ over bo^s. It is not hard to see that $R(8s+8) \to bo \wedge R(8s) \approx bo^{4s}$ lifts to bo^{4s+3}, so that we can extend to a map $bo \wedge R(8s+8) \to bo^{4s+3}$; likewise, $DR(8s-8) \to bo \wedge DR(8s) \approx bo^{-4s}$ lifts to bo^{-4s+3}, so we can extend to a map $bo \wedge DR(8s-8) \to bo^{-4s+3}$. The resulting tower is an Adams tower. □

Here is a chart which presents the Adams E_2-term for $bo \wedge DR(16)$.



A chart for $\mathrm{Ext}^{s,t}_{A(1)}(H^*(DR(16)), \mathbb{Z}/2)$ for $s < 16$.

7. Finite localization

There exists a functor L_n^f and natural map $X \to L_n^f X$, called **finite localization**. It is Bousfield localization with respect to the wedge $\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)$, where $\mathrm{Tel}(k)$ denotes the v_k-telescope on some chosen type k finite complex. The functor L_n^f is characterized by the following properties [11].

1. The fiber $C_n^f X = \mathrm{fib}(X \to L_n^f X)$ is a homotopy colimit of some diagram of type-$(n+1)$ finite complexes.
2. There are no essential maps from a type $(n+1)$ finite complex to $L_n^f X$ (i.e., $L_n^f X$ is L_n^f**-local**).
3. $L_n^f X$ is **smashing**; i.e., $L_n^f X \approx X \wedge L_n^f S^0$.

Note also that 1 and 2 imply that $L_n^f X \approx X$ if X is L_n^f-local. Also, if X is a type n finite complex, then $L_n^f X \approx v^{-1} X$, where $v \colon \Sigma^d X \to X$ is a v_n-self map of X.

Theorem 7.1. *Let X be an fp-spectrum with* $\mathrm{fptype}(X) \leq n$*, and let $\{X_s\}$ be a $\mathbb{Z}$-indexed Adams tower for X, with colimit $\widehat{X}$. Then $L_n^f X \approx \widehat{X}$.*

Proof. Consider the map $i \colon X \to \widehat{X}$. It suffices to show that

1. $L_n^f(i)$ is an equivalence, and
2. $\widehat{X}$ is L_n^f-local.

Claim 1 follows immediately from the fact that $L_n^f H\mathbb{F}_p \approx *$ and that L_n^f is smashing, since $\widehat{X}$ is obtained from X by attaching mod p Eilenberg-Mac Lane spectra.

To prove claim 2, we must show that $\widehat{X}$ admits no essential maps from a type-$(n+1)$ finite complex F, or equivalently, that $\widehat{X} \wedge F \approx *$ for any such complex. This follows from the following:

3. For any finite complex F, $\{X_s \wedge F\}$ is a $\mathbb{Z}$-indexed Adams tower for $X \wedge F$, and $\widehat{X} \wedge F \approx \operatorname{hocolim} X_{-s} \wedge F$.
4. If Y is an fp-spectrum with $\pi_* Y$ finite, then $\widehat{Y} \approx *$ for any $\mathbb{Z}$-indexed Adams tower for Y.

The proof of claim 3 is straightforward.

To prove claim 4, apply Brown-Comenetz duality to the sequence

$$Y = Y_0 \to Y_{-1} \to Y_{-2} \to \dots \to \widehat{Y}.$$

This produces a tower

$$I\widehat{Y} \to \dots \to IY_{-2} \to IY_{-1} \to IY_0 = IY$$

which is easily seen to be an Adams tower for IY by (5.5). Since IY is connective and p-complete and $\pi_* IY$ is finite, its Adams spectral sequence converges, and $I\widehat{Y} \approx *$, whence $\widehat{Y} \approx *$. □

7.2. **Relation between L_n^f and L_n.** Let $X \to L_n X$ denote Bousfield localization with respect to $K(0) \vee \dots \vee K(n)$. There is a natural map $t_n \colon L_n^f X \to L_n X$; this map is an equivalence for $X \approx S^0$, and hence for all X, if and only if the Telescope Conjecture holds for all $m \leq n$. This conjecture is true for $n = 0$ and $n = 1$, and is believed to be false for $n \geq 2$.

It is reasonable to ask whether t_n is an equivalence when X is an fp-spectrum. We note that t_n is an equivalence in the following cases, which include all cases we know of.

1. Since $BP\langle n\rangle$ is a BP-module spectrum, one can compute $L_n BP\langle n\rangle$ using the chromatic tower method of [14, Sec. 6]. (We would like to thank Hal Sadofsky for pointing this out to us.) This calculation shows in particular that the fiber of $BP\langle n\rangle \to L_n BP\langle n\rangle$ is coconnective with torsion homotopy; thus the fiber is killed by L_n^f, and hence t_n is an equivalence on $L_n BP\langle n\rangle$, and is in fact an equivalence on the thick subcategory of $\mathcal{C}$ generated by $BP\langle n\rangle$.
2. For any fp-spectrum X obtained by the procedure of the proof of (3.5) the map $t_n \colon L_n^f X \to L_n X$ is an equivalence, since by construction the fiber of $X \to L_n X$ is coconnective with torsion homotopy.

We make the following conjecture.

Conjecture 7.3. *The map $t_n \colon L_n^f X \to L_n X$ is an equivalence for all fp-spectra X.*

This conjecture is of interest, because it would give information on how badly the Telescope Conjecture fails, assuming it does fail. Namely, suppose F is a type n finite complex and $v^{-1}F$ is its v_n-telescope; then if the Telescope Conjecture fails for n and Conjecture (7.3) holds, it follows that $\pi_k(v^{-1}F)$ is an infinite group for some $k \in \mathbb{Z}$, whereas $\pi_k L_n F$ is finite for all k. To see this, we argue as follows; if each $\pi_k(v^{-1}F)$ were a finite group, then the proof of (3.5) would apply to show that there exists an fp-spectrum X with $L_n^f X \approx v^{-1}F$. (We could take X to be a connective cover of $v^{-1}F$, for example.) However, Conjecture (7.3) would then imply that $v^{-1}F \approx L_n X \approx L_n F$, contradicting the failure of the Telescope Conjecture.

Conjecture (7.3) also would imply, using (8.9) of the next section, that the two conjectures (3.8) and (3.9) discussed in section 3 are in fact equivalent statements.

8. Duality

8.1. **Duality for fp-spectra of type less than n.** Let

$$W_n X = I C_n^f X,$$

where $C_n^f = \mathrm{fib}(X \to L_n^f X)$. Thus W_n is a contravariant functor from spectra to spectra. Since L_n^f is smashing,

$$W_n X \approx \mathcal{F}(X \wedge C_n^f S^0, IS^0) \approx \mathcal{F}(X, W_n S^0).$$

That is, $W_n S^0$ is a "dualizing complex" for W_n. Furthermore, there is a natural map $X \to W_n W_n X$; this map is adjoint to the evaluation map $X \wedge \mathcal{F}(X, W_n S^0) \to W_n S^0$.

Note that W_n vanishes on L_n^f-local spectra. Also, if $\pi_* X$ is finite, then $L_n^f X \approx *$ and we have $W_n X \approx IX$.

Recall that $\mathcal{C}_n$ denotes the homotopy category of fp-spectra with fp-type $\leq n$.

Theorem 8.2. *Let X be an object in $\mathcal{C}_n$.*

1. *There is a natural isomorphism $H_* W_n X \approx \tilde{I}(H_* X)$.*
2. *$W_n X$ is in $\mathcal{C}_n$.*
3. *The natural map $X \to W_n W_n X$ is an equivalence.*

From this we obtain the following.

Corollary 8.3. *The functor W_n restricts to a functor $W_n \colon \mathcal{C}_n^{\mathrm{op}} \to \mathcal{C}_n$, which is an equivalence of categories.*

Proof. The corollary follows from the fact that the functor W_n is "self-adjoint"; that is,

$$W_n \colon \mathcal{S}^{\mathrm{op}} \rightleftarrows \mathcal{S} : (W_n)^{\mathrm{op}}$$

is a pair of adjoint functors, where $\mathcal{S}$ represents the homotopy category of spectra. Part 2 of (8.2) says that W_n carries $\mathcal{C}_n^{\mathrm{op}}$ into $\mathcal{C}_n$, and part 3 says that the restriction of W_n to $\mathcal{C}_n$ gives an adjoint equivalence $W_n \colon \mathcal{C}_n^{\mathrm{op}} \rightleftarrows \mathcal{C}_n : (W_n)^{\mathrm{op}}$ of categories. ☐

Proof of Theorem 8.2. If X is an fp-spectrum, choose a $\mathbb{Z}$-indexed Adams tower $\{X_s\}$, and as in (6.1) write $X_q^p = \mathrm{cof}(X_{p+1} \to X_q)$. Then

$$C_n^f X \approx \Sigma^{-1} \operatorname*{hocolim}_{s \to \infty} X_{-s}^{-1}$$

by (7.1), and thus

$$W_n X \approx \Sigma \left(\operatorname*{holim}_{s \to \infty} I X_{-s}^{-1} \right). \tag{8.4}$$

Write $H_* X \approx A_* \otimes_{A_*(n)} M$. We can construct the bottom part of the $\mathbb{Z}$-indexed tower to realize any projective resolution of M by finite $A_*(n)$-modules, e.g., by the resolution dual to the minimal resolution $0 \to \check{M} \to C_\bullet$ of $\check{M}$ by finite $A_*(n)$-modules. Then (8.4) immediately implies that $W_n X$ is connective and p-complete, since the tower $\{I(X_{-s}^{-1})\}_{s \geq 1}$ must necessarily be the tower associated with an

Adams tower for W_nX which realizes $C_\bullet$, and so the connectivity of $I(X_{-s}^{-1})$ is bounded below by a fixed N for all $s \geq 1$.

Note that for any Y we have that $H\mathbb{F}_p \wedge L_n^f Y \approx *$, whence the map $H_*C_n^f Y \to H_*Y$ is an isomorphism, so that there exists by (5.5) a natural comparison map

$$\iota^n(Y)\colon \tilde{I}(H_*Y) \xrightarrow{\sim} \tilde{I}(H_*C_n^f Y) \to H_*W_nY.$$

We want to show that $\iota^n(X)$ is an isomorphism.

Since X has a $\mathbb{Z}$-indexed tower, there is a sequence

$$\ldots \to HV_{-2} \to HV_{-1} \to X$$

which induces an exact sequence

$$H_*HV_{-2} \to H_*HV_{-1} \to H_*X \to 0$$

on homology. Applying W_n to this sequence gives a sequence

$$W_nX \to W_nHV_{-1} \to W_nHV_{-2} \to \ldots$$

which corresponds to an Adams resolution for W_nX and hence induces an exact sequence

$$0 \to H_*W_nX \to H_*W_nHV_{-1} \to H_*W_nHV_{-2}.$$

It is clear that the comparison maps $\iota^n(HV_{-s})$ are isomorphisms, and so $\iota^n(X)$ is an isomorphism, since $\tilde{I}$ is exact. Thus we have proved parts 1 and 2 of the proposition.

Part 3 follows from parts 1 and 2, together with (8.7), to be proved below. □

Remark 8.5. If X is a spectrum which might not be p-complete, and X_p denotes its p-completion, then one can show via an arithmetic square argument that $C_n^f X \approx C_n^f X_p$, and hence $W_nX \approx W_nX_p$. In particular, if X_p is an fp-spectrum, we may conclude that $W_nW_nX \approx X_p$.

We still owe the reader one more fact.

Proposition 8.6. *Suppose X and W_nX have finitely presented homology. Then there is a commutative square*

$$\begin{array}{ccc} \tilde{I}(H_*W_nX) & \xrightarrow{\iota^n(W_nX)} & H_*W_nW_nX \\ \Big\downarrow{\scriptstyle \tilde{I}\iota^n(X)} & & \Big\uparrow \\ \tilde{I}\tilde{I}(H_*X) & \xleftarrow{\sim} & H_*X \end{array}$$

Corollary 8.7. *If X and W_nX have finitely presented homology and the maps*

$$\iota^n(X)\colon \tilde{I}(H_*X) \to H_*W_nX, \qquad \text{and} \qquad \iota^n(W_nX)\colon \tilde{I}(H_*W_nX) \to H_*W_nW_nX$$

are isomorphisms, then the the map $X \to W_nW_nX$ induces an isomorphism in homology.

Proof of Proposition 8.6. Choose resolutions $X \to HC_0 \to HC_1$ and $W_nX \to HD_0 \to HD_1$. This leads to a sequence

$$H\check{D}_1 \to H\check{D}_0 \to W_nW_nX \to HC_0 \to HC_1.$$

The diagonal map in the diagram is the induced map

$$\operatorname{Cok}\left(H_*H\check{D}_1 \to H_*H\check{D}_0\right) \to \operatorname{Ker}\left(H_*HC_0 \to H_*HC_1\right).$$

It is straightforward to check commutativity of the diagram. □

8.8. **A finiteness result.**

Proposition 8.9. *Let X be an fp-spectrum with* $\operatorname{fptype}(X) = n$*, and let* $Y = L_n^f X$ *be its finite localization. Then for each* $k \in \mathbb{Z}$ *the homotopy group* $\pi_k Y$ *has the form*

$$\pi_k Y \approx F_k \oplus \mathbb{Z}_p^{\oplus a_k} \oplus (\mathbb{Q}/\mathbb{Z}_{(p)})^{\oplus b_k} \oplus \mathbb{Q}_p^{\oplus c_k},$$

where F_k is a finite p-group, $a_k = 0 = c_k$ for all sufficiently small $k \ll 0$, and $b_k = 0 = c_k$ for all sufficiently large $k \gg 0$.

Proof. There is a fiber sequence $C_n^f X \to X \to L_n^f X$, and the Brown-Comenetz dual $W_n X$ of $C_n^f X$ is an fp-spectrum by (8.2). Thus X is connective with $\pi_k X \approx F_k \oplus \mathbb{Z}_p^{m_k}$, and $C_n^f X$ is coconnective with $\pi_k C_n^f X \approx F_k' \oplus (\mathbb{Q}/\mathbb{Z}_{(p)})^{n_k}$. The image and coimage of the connecting map $\pi_k C_n^f X \to \pi_k X$ can only be a finite torsion group; thus, to prove the result for $\pi_k L_n^f X$ we need to show that, in a group extension of the form

$$0 \to F \oplus \mathbb{Z}_p^m \to M \to F' \oplus (\mathbb{Q}/\mathbb{Z}_{(p)})^n \to 0,$$

where F and F' are finite p-groups, that M is as described in the statement of the proposition.

It is easy to reduce to the case when $F = 0 = F'$. Then extensions are classified by elements of $\operatorname{Ext}((\mathbb{Q}/\mathbb{Z}_{(p)})^n, \mathbb{Z}_p^m) \approx \hom(\mathbb{Z}_p^n, \mathbb{Z}_p^m)$; if $A \in \hom(\mathbb{Z}_p^n, \mathbb{Z}_p^m)$ classifies the extension then

$$M \approx \operatorname{Cok}\left(\mathbb{Z}_p^n \xrightarrow{(A,I)} \mathbb{Z}_p^m \oplus \mathbb{Q}_p^n\right)$$

where $I\colon \mathbb{Z}_p^n \to \mathbb{Q}_p^n$ is the standard inclusion. It follows that $M \approx \mathbb{Z}_p^m / \operatorname{im} A \oplus \mathbb{Q}_p^n / \ker A$; this can be shown by choosing a map $B\colon \mathbb{Z}_p^m \to \mathbb{Q}_p^n$ such that $(I - BA)\colon \mathbb{Z}_p^n \to \mathbb{Q}_p^n$ projects to the kernel of $A \otimes \mathbb{Q}$, in which case there is an exact sequence

$$0 \to \mathbb{Z}_p^n \xrightarrow{(A,I)} \mathbb{Z}_p^m \oplus \mathbb{Q}_p^n \xrightarrow{(x,y)\to(x,y-Bx)} \mathbb{Z}_p^m / \operatorname{im} A \oplus \mathbb{Q}_p^n / \ker A \to 0$$

which realizes the splitting of M. Now the result follows from the fact that $\mathbb{Z}_p^m / \operatorname{im} A \approx F \oplus \mathbb{Z}_p^a$ and $\mathbb{Q}_p^n / \ker A \approx (\mathbb{Q}/\mathbb{Z}_{(p)})^b \oplus \mathbb{Q}_p^c$, where F is a finite p-group. ☐

8.10. **Duality for fp-spectra.** Recall that there exist natural transformations $L_{n+1}^f X \to L_n^f X$, and hence natural transformations $W_n X \to W_{n+1} X$. We define

$$WX = \operatorname*{hocolim}_{n \to \infty} W_n X.$$

If X is an fp-spectrum with $\operatorname{fptype}(X) = n$, then (7.1) shows that $L_m^f X \approx L_n^f X$ for $m \geq n$, and thus $W_m X \approx W_n X$ for $m \geq n$. This, together with (8.3) and the fact that $\mathcal{C} = \bigcup_n \mathcal{C}_n$ is the homotopy category of all fp-spectra, gives

Theorem 8.11. *The functor W induces an equivalence of categories $W\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, and $H_* WX \approx \tilde{I}(H_* X)$ for all X in $\mathcal{C}$.*

9. Calculations

In this section we compute WX in several cases, and thus implicitly compute $L_n^f X$ (and $L_n X$, by (7.2)) for sufficiently large n.

Lemma 9.1. *If X is a ring spectrum and Y is an X-module spectrum, then $W_n Y$ is also an X-module spectrum.*

Proof. This is a formal consequence of the fact that $W_n Y \approx \mathcal{F}(Y, W_n S^0)$. ☐

Proposition 9.2. *If X is an fp-spectrum which is a ring spectrum, H_*X is self-dual as a finitely-presented comodule (i.e., $\tilde{I}(H_*X) \approx \Sigma^d H_*X$ for some d), and H_*X does not split over the Steenrod algebra, then $WX \approx \Sigma^d X$.*

Proof. Choose a map $S^d \to WX$ which hits the bottom homology class, dual to the unit in H_*X. By (9.1) this map extends to a map $\Sigma^d X \to WX$ of X-module spectra, and this map is necessarily an isomorphism on mod p homology, and hence an equivalence. □

Corollary 9.3. *We have that*

1. $Wk(n) \approx \Sigma^{2p^n-1} k(n)$,
2. $WBP\langle n\rangle \approx \Sigma^{e(n)} BP\langle n\rangle$, *where* $e(n) = 2\frac{p^{n+1}-1}{p-1} - (n+1)$,
3. $Wbu \approx \Sigma^4 bu$ *(at all primes),*
4. $Wbo \approx \Sigma^6 bo$ *(at all primes), and*
5. $Weo_2 \approx \Sigma^{23} eo_2$ *at* $p = 2$ *and at* $p = 3$.

Proof. The only case which needs comment is 5. In this case it can be derived from the following facts [5]. At $p = 2$, $eo_2 \wedge F \approx BP\langle 2\rangle$, where F is a certain finite complex with $H^*F \approx \mathcal{D}A^*(1)$, the "double" of $A^*(1)$. At $p = 3$, $eo_2 \wedge (S^0 \cup_\alpha e^4 \cup_{2\alpha} e^8) \approx BP\langle 2\rangle \vee \Sigma^8 BP\langle 2\rangle$. □

Remark 9.4. In each of the above examples, we can read off $\pi_* L_n X$ from our knowledge of the homotopy of X. In particular, in cases 2–5 there is a wide "gap" between the first copy of $\mathbb{Z}_p$ and the last copy of $\mathbb{Q}/\mathbb{Z}_{(p)}$ in the homotopy of $L_n X$; if $WX \approx \Sigma^d X$, then

$$\pi_s X \approx \begin{cases} \mathbb{Z}_p & \text{if } s = 0, \\ 0 & \text{if } 1-d < s < 0, \text{ and} \\ \mathbb{Q}/\mathbb{Z}_{(p)} & \text{if } s = 1-d. \end{cases}$$

There is a convenient heuristic for reading off the expected size of the "gap" in $\pi_* L_n X$ for many ring spectra of the above type. If $\pi_* X \otimes \mathbb{Q} \approx \mathbb{Q}_p[x_1, \ldots, x_n]$, then the size of the gap is $\sum_{i=1}^n (|x_i| + 1)$. For example, $\pi_* eo_2 \otimes \mathbb{Q} \approx \mathbb{Q}_p[x_8, x_{12}]$ at $p = 2$ or 3, so the gap is $(8+1) + (12+1) = 22$. For $BP\langle n\rangle$ the gap is the same as the dimension of the Toda complex $V(n)$, should it exist.

Recall from (4.10) that if $H_*X \approx A_* \otimes_{A_*(n)} M$, then $\tilde{I}(H_*X) \approx A_* \otimes_{A_*(n)} \Sigma^d \check{M}$, where d is the dimension of the "top cell" of $A_*(n)$.

Proposition 9.5. *Let J_p denote the connective image-of-J spectrum completed at the prime p.*

1. *For p odd,* $WJ_p \approx \Sigma^3 J_p$.
2. *For $p = 2$, there is a cofiber sequence* $\Sigma^3 H\mathbb{F}_2 \to J_2 \wedge (S^0 \cup_2 e^1 \cup_\eta e^3) \to WJ_2$.

Proof. At an odd prime, J_p is the fiber of any map $BP\langle 1\rangle \to \Sigma^q BP\langle 1\rangle$, $q = 2(p-1)$, which sends the cohomology generator $\iota \in H^q \Sigma^q BP\langle 1\rangle$ to $P^1 \iota \in H^q BP\langle 1\rangle$. At $p = 2$, J_2 is the fiber of any map $bo \to bspin$ which sends the cohomology generator $\iota_{bspin} \in H^4 bspin$ to $Sq^4 \iota_{bo} \in H^4 bo$ [10]. In either case, the map in cohomology is induced from a map of $A^*(2)$-modules.

We leave the odd prime case to the reader. Suppose $p = 2$, and let $F = S^0 \cup_2 e^1 \cup_\eta e^3$. Since $bspin \approx \Sigma^7 bo \wedge DF$ and $H^*(F \wedge DF)$ is a direct sum over the

Steenrod algebra of a spherical class in dimension 0 with a free $A^*(1)$ module on one generator, we see that $J \wedge F$ fits in a fiber sequence

$$J \wedge F \to bo \wedge F \to \Sigma^7 bo \vee \Sigma^4 H\mathbb{F}_2.$$

We can kill the copy of $H\mathbb{F}_2$ by taking the evident cofiber $\Sigma^3 H\mathbb{F}_2 \to J \wedge F \to C$, so that we obtain a cofiber sequence

$$C \to bo \wedge F \to \Sigma^7 bo.$$

On applying W we get a fiber sequence

$$\Sigma^{-1} bo \to \Sigma^6 bo \wedge DF \to WC \to bo \xrightarrow{f} \Sigma^7 bo \wedge DF.$$

We can compute the action on f on cohomology, since by (4.10) it is induced from a map of $A^*(2)$-modules. The computation, which is straightforward, shows that the bottom class of $\Sigma^7 bo \wedge DF$ hits $Sq^4 \iota_{bo} \in H^4 bo$, and thus $WC \approx J$. □

Corollary 9.6 (Hopkins)**.**

$$I(L_1 S^0) \approx L_1(S^{-1} \cup_2 e^0 \cup_\eta e^2).$$

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Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: mark@math.nwu.edu
E-mail address: rezk@math.nwu.edu