Cartesian presentations of weak n-categories

An introduction to Θ_n-spaces

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Idea of higher category theory

- An ∞-**category** is a gadget equipped with
  - objects,
  - 1-morphisms between objects,
  - 2-morphisms between 1-morphisms,
  - 3-morphisms between 2-morphisms,
  - etc.
- An (∞, n)-**category** is one such that all k-morphisms are "invertible", for $k > n$.

Goal of this talk

I want to discuss an approach to (∞, n)-categories, based on the following ideas:

- An $(\infty, 0)$-category ($=$ an ∞-groupoid) is a **space**.
  ("Homotopy hypothesis".)
- An (∞, n)-category should be more-or-less the same thing as a **category enriched over** $(\infty, n-1)$**-categories**.
- The collection of (∞, n)-categories should have internal function objects,
  i.e., (∞, n)-categories should be **Cartesian closed**,
  and thus be an example of some kind of $(\infty, n+1)$-category.
- We should avoid interpreting the above ideas too **strictly**.

Desired features of $(\infty, 1)$-categories

Let $\mathrm{Cat}_{\infty,1}$ = "category" of $(\infty, 1)$-categories.

- **equivalences**: class of morphisms in $\mathrm{Cat}_{\infty,1}$
- $\mathrm{Cat}_{\infty,1}$ is Cartesian closed:
  $C, D \in \mathrm{Cat}_{\infty,1} \implies \{C, D\}$, right adjoint to $\times$
- $\mathrm{Gpd}_{\infty} \subset \mathrm{Cat}_{\infty,1}$ full subcategory of ∞-groupoids
  $C^{\mathrm{gpd}} \subseteq C$ maximal sub-∞-groupoid of C
- classifying space functor $B \colon \mathrm{Gpd}_{\infty} \to \mathrm{Sp}$:

$$\{\text{groupoids up to equivalence}\} \iff \{\text{spaces up to weak equivalence}\}$$

Can we understand $\mathrm{Cat}_{\infty,1}$ using spaces?

Given $C \in \mathrm{Cat}_{\infty,1}$, let

$$\mathcal{F} = \mathcal{F}_C \colon \mathrm{Cat}_{\infty,1}^{\mathsf{op}} \to \mathrm{Sp}$$
$$A \mapsto B(\{A, C\}^{\mathrm{gpd}}) = \mathrm{Map}(A, C)$$

(representable space valued presheaf on $\mathrm{Cat}_{\infty,1}$)

- Think of $\mathcal{F}_C(\bullet) = B(C^{\mathrm{gpd}})$ as the "moduli space" of objects of C:

$$B(C^{\mathrm{gpd}}) \approx \coprod_{\substack{[X] \\ \text{iso. classes}}} B\mathrm{Aut}(X).$$

  ($\bullet =$ "freestanding object" category)
- Think of $\mathcal{F}_C(A)$ as the "moduli space" of functors $A \to C$
- $\mathrm{Cat}_{\infty,1} \iff \{$representable presheaves in $\mathrm{Psh}(\mathrm{Cat}_{\infty,1}, \mathrm{Sp})\}$
  Yoneda lemma!

Example: C = finite sets

C = category of finite sets

- "Size" is a complete isomorphism invariant of finite sets
  $\mathrm{Aut}(\{1, \ldots, n\}) = \Sigma_n$ symmetric group

$$\mathcal{F}_C(\bullet) \approx \coprod_{[S]} B\mathrm{Aut}(S) \approx \coprod_{n \geq 0} B\Sigma_n$$

Example: C = finite sets, continued

- Let $[1] = (\bullet \to \bullet)$
- $\{[1], C\}$ = category of functors $[1] \to C$
  Objects: morphisms $f\colon S_0 \to S_1$ in C
  Morphisms: commutative diagrams

$$\begin{array}{ccc} S_0 & \overset{\sim}{\to} & T_0 \\ \downarrow & & \downarrow \\ S_1 & \underset{\sim}{\to} & T_1 \end{array}$$

- C = finite sets
  $p(f) = (p_0, p_1, p_2, \ldots)$ where $p_k = \#$ of fibers of f with size k

$$\mathcal{F}_C([1]) \approx \coprod_{[S_0 \xrightarrow{f} S_1]} B\mathrm{Aut}(S_0 \xrightarrow{f} S_1) \approx \coprod_{\underline{p}} B\left(\prod_k \Sigma_k \wr \Sigma_{p_k}\right)$$

- If f is isomorphism, $p(f) = (0, n, 0, 0, \ldots)$, so $B\mathrm{Aut}(f) \approx B\Sigma_n$

General properties of $\mathcal{F}_C$

- $\mathcal{F}_C([1])_{\text{inv}} \stackrel{\text{def}}{=}$ subspace of $\mathcal{F}_C([1])$ of path components containing invertible maps
  $\mathcal{F}_C(\bullet) \to \mathcal{F}_C([1])$ factors through a weak equivalence
  $$\mathcal{F}_C(\bullet) \xrightarrow{\sim} \mathcal{F}_C([1])_{\text{inv}} \subseteq \mathcal{F}_C([1]).$$
- $\mathcal{F}_C(A)$ can always be recovered as a homotopy limit from diagrams involving the spaces $\mathcal{F}_C(\bullet)$ and $\mathcal{F}_C([1])$.
  For instance
  $$\mathcal{F}_C(0 \to 1 \to 2) \approx \lim\big(\mathcal{F}_C(0 \to 1) \to \mathcal{F}_C(1) \leftarrow \mathcal{F}_C(1 \to 2)\big)$$
  and similarly for $\mathcal{F}_C(0 \to 1 \to \cdots \to n)$.

Recovering C from $\mathcal{F}$

$\Delta \subset \mathrm{Cat}$: full subcategory of categories of the form

$$[m] = (0 \to 1 \to 2 \to \cdots \to m)$$

Can recover C, up to equivalence, from the restriction of $\mathcal{F}_C$ to Δ:

- $\pi_0 \mathcal{F}_C([0])$ = isomorphism classes of objects of C
- $\mathrm{Map}_C(X, Y) \approx \mathrm{hofiber}_{(X,Y)}\left[\mathcal{F}_C([1]) \to \mathcal{F}_C([0]) \times \mathcal{F}_C([0])\right]$
- composition is defined using

$$\mathrm{Map}_C(X, Y) \times \mathrm{Map}_C(Y, Z) \approx \\ \mathrm{hofiber}_{(X,Y,Z)}\left[\mathcal{F}_C([2]) \to \mathcal{F}_C([0]) \times \mathcal{F}_C([0]) \times \mathcal{F}_C([0])\right]$$

- associativity of composition uses fibers of $\mathcal{F}_C([3]) \to \mathcal{F}_C([0])^4$

Complete Segal space: a functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Sp}$ satisfying the following.

- **Segal condition.** For all $k \geq 2$,

$$X([k]) \xrightarrow{\sim} \lim \left(\begin{array}{ccccccc} X([1]) & & X([1]) & & \cdots & & X([1]) \\ & \searrow \; X[0] \; \swarrow & & \searrow \; \cdots \; \swarrow & & \searrow \; X[0] \; \swarrow & \end{array} \right)$$

- **Completeness condition.**
  The map $X([0]) \to X([1])$ factors through a weak equivalence $X([0]) \to X([1])_{\mathrm{inv}} \subseteq X([1])$.

(If $X \in \mathrm{Psh}(\Delta, \mathrm{Sp})$ satisfies the Segal condition,
$X([1])_{\mathrm{inv}} \overset{\mathrm{def}}{=}$ union of components of $X([1])$ which contain elements invertible in the "homotopy category" of X.)

Complete Segal spaces and $(\infty, 1)$-categories

A complete Segal space X has

- "objects" $\iff$ points of $X([0])$
- "morphism spaces" for $a, b \in X([0])$

$$\mathrm{MAP}_X(a, b) \stackrel{\mathrm{def}}{=} \mathrm{hofiber}_{(a,b)}\big[X([1]) \to X([0]) \times X([0])\big].$$

- a weakly defined "composition"

Theorem (Bergner)

$$\{\textit{complete Segal spaces}\} \iff \{\textit{categories enriched over spaces}\}.$$

That is:

$$\{\textit{complete Segal spaces}\} \iff \{\textit{categories enriched over } (\infty, 0)\textit{-categories}\}.$$

Also equivalent to: **Segal categories** (Bergner), **quasicategories** (Joyal-Tierney).

Presentations

Definition

- A **presentation** $(C, \mathcal{S})$ consists of
  - $C =$ small category,
  - $\mathcal{S} = \{s \colon S \to S'\} =$ set of morphisms in $\mathrm{Psh}(C, \mathrm{Sp})$.
- An $\mathcal{S}$-**local presheaf** is $X \in \mathrm{Psh}(C, \mathrm{Sp})$ such that for all $s \in \mathcal{S}$,

$$\mathrm{Map}(s, X) \colon \mathrm{Map}(S', X) \to \mathrm{Map}(S, X)$$

  is weak equivalence of spaces. ($\mathrm{Map} =$ derived mapping space.)
- $\mathrm{Psh}(C, \mathrm{Sp})_{\mathcal{S}} \overset{\mathrm{def}}{=}$ full subcategory of $\mathcal{S}$-local presheaves in $\mathrm{Psh}(C, \mathrm{Sp})$.
- $\overline{\mathcal{S}} \overset{\mathrm{def}}{=}$ class of maps:
  $f \in \overline{\mathcal{S}}$ iff $\mathrm{Map}(f, X)$ is a weak equivalence for all $\mathcal{S}$-local X
  (sometimes called $\mathcal{S}$-**local equivalences**, or **saturation of** $\mathcal{S}$.)

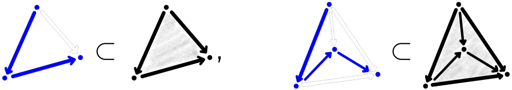
Note: $h\mathrm{Psh}(C, \mathrm{Sp})_{\mathcal{S}} \approx h\mathrm{Psh}(C, \mathrm{Sp})[\overline{\mathcal{S}}^{-1}]$.

Presentation of complete Segal spaces

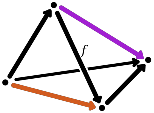
Complete Segal spaces are presented by $(\Delta, \mathcal{S})$, where $\mathcal{S}$ consists of

$$\mathrm{se}_k \colon G[k] \to F[k] \quad (\text{for } k \geq 2), \qquad \mathrm{cp} \colon Z \to F[0].$$

- $F[k]$ = presheaf represented by $[k] \in \mathrm{ob}\Delta$
- $G[k] \subset F[k]$, e.g.:

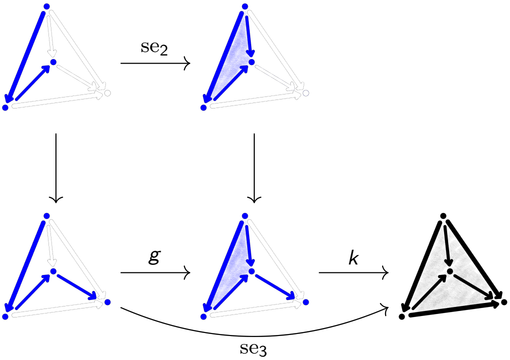


- $Z = F[3]/\sim \; = \mathrm{colim}(F[3] \leftarrow F[1] \amalg F[1] \to F[0] \amalg F[0])$.



$$\mathrm{Map}(Z, X) \approx X([1])_{\mathrm{inv}} \subseteq X([1]) \text{ if } X \text{ satisfies Segal condition}$$

An example of elements of $\overline{\mathcal{S}}$.



- $\mathrm{se}_2 \in \overline{\mathcal{S}} \implies g \in \overline{\mathcal{S}}$
- $g, \mathrm{se}_3 \in \overline{\mathcal{S}} \implies k \in \overline{\mathcal{S}}$

Cartesian presentations

- $\mathrm{Psh}(C, \mathrm{Sp})$ is **Cartesian closed**: internal function object $\{X, Y\}$.

$$X \to \{Y, Z\} \quad \iff \quad X \times Y \to Z.$$

- In what follows, $\{X, Y\}$ = the *derived* version of function object.

Definition

A presentation $(C, \mathcal{S})$ is **Cartesian** if for all $X \in \mathrm{Psh}(C, \mathrm{Sp})$,

$$Y \in \mathrm{Psh}(C, \mathrm{Sp})_{\mathcal{S}} \quad \Longrightarrow \quad \{X, Y\} \in \mathrm{Psh}(C, \mathrm{Sp})_{\mathcal{S}}.$$

- $(C, \mathcal{S})$ Cartesian $\Longrightarrow$
  $\mathrm{Psh}(C, \mathrm{Sp})_{\mathcal{S}}$ has a (derived) internal function object,
  which is **computed** as the function object between the underlying presheaves.

Theorem (R.)

The presentation $(\Delta, \mathcal{S})$ defining complete Segal spaces is Cartesian.

- To show that a presentation $(C, \mathcal{S})$ is Cartesian, check:

$$(S \xrightarrow{s} S') \in \mathcal{S} \qquad \Longrightarrow \qquad (S \times Fc \xrightarrow{s \times \mathrm{id}} S' \times Fc) \in \overline{\mathcal{S}}$$

  for all $c \in \mathrm{ob}C$
  $Fc =$ presheaf represented by c

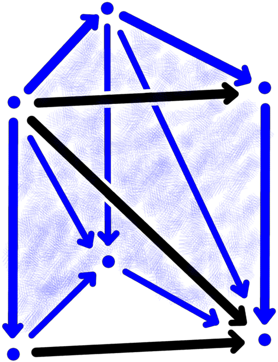
- To prove the Theorem, show that

$$G[k] \times F[m] \xrightarrow{\mathrm{se}_k \times \mathrm{id}} F[k] \times F[m], \qquad Z \times F[m] \xrightarrow{\mathrm{cp} \times \mathrm{id}} F[0] \times F[m]$$

  are in $\overline{\mathcal{S}}$.

Idea of the proof

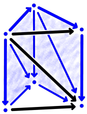
Consider $F[2] \times F[1] \supset G[2] \times F[1]$



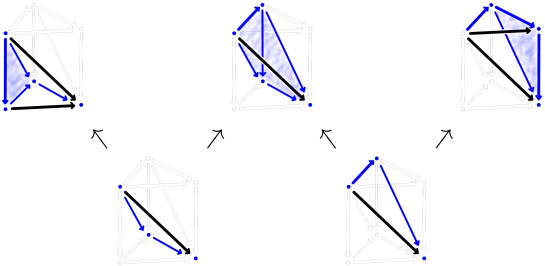
Want to show: X complete Segal space $\Longrightarrow$
$\mathrm{Map}(F[2] \times F[1], X) \to \mathrm{Map}(G[2] \times F[1], X)$ is a weak equivalence

Idea of the proof, (continued)

$$F[2] \times F[1] =$$



$$= \operatorname{colim}\left(F[3] \leftarrow F[2] \rightarrow F[3] \leftarrow F[2] \rightarrow F[3]\right).$$



$\mathrm{Map}(\text{Black\&Blue}, X) \rightarrow \mathrm{Map}(\text{Blue}, X)$ is a weak equivalence.
if X is a complete Segal space

We want to base a definition of (∞, n)-categories on the following principles (here $C, D \in \mathrm{Cat}_{\infty,n}$):

- function objects $\{C, D\} \in \mathrm{Cat}_{\infty,n}$
- maximal sub-∞-groupoid $C^{\mathrm{gpd}} \subseteq C$
- ∞-groupoids are spaces
- these constructions invariant under equivalence

$\Longrightarrow$ functor

$$\begin{aligned} \mathcal{F} = \mathcal{F}_C \colon \mathrm{Cat}_{\infty,n}^{\mathsf{op}} &\to \mathrm{Sp} \\ A &\mapsto \{A, C\}^{\mathrm{gpd}} \approx \mathrm{Map}(A, C) \end{aligned}$$

- To make this concrete, need a suitable small subcategory of $\mathrm{Cat}_{\infty,n}$

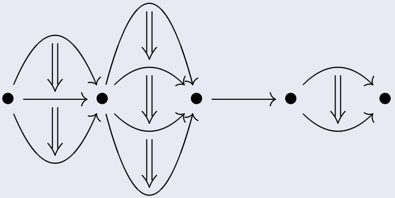
The category Θ_n

Θ_n introduced by Joyal; related to Batanin's "pasting diagrams"

Θ_n is to n-categories as $\Delta = \Theta_1$ is to 1-categories

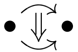
Definition (Vague)

Θ_n is the full subcategory of strict n-categories consisting of objects which "look like"



The name of this object (of Θ_2) is $[4]([2],[3],[0],[1])$.

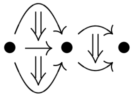
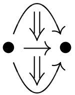
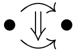
k-cells in Θ_n for $0 \leq k \leq n$. Notation:

$O_0 = (\bullet)$, $O_1 = (\bullet \to \bullet)$, $O_2 = \Bigg($  $\Bigg)$, $\ldots$

Idea of Θ_n-spaces

A Θ_n**-space** is a functor $X\colon \Theta_n^{\mathrm{op}} \to \mathrm{Sp}$ satisfying

- **Segal conditions.** $X(\theta) =$ homotopy limit of $X(O_k)$'s:

X () $\approx \lim \Big[X$ () $\to X(\bullet) \leftarrow X$ () $\Big]$

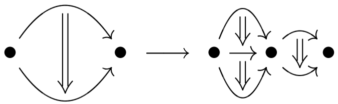
$$\approx \lim \begin{bmatrix} X(O_2) & & & & \\ \downarrow & \searrow & & & \\ X(O_1) & \to & X(O_0) & \leftarrow & X(O_2) \\ \uparrow & \nearrow & & & \\ X(O_2) & & & & \end{bmatrix}$$

- **Completeness conditions.** $X(O_{k-1}) \to X(O_k)$ factors through a weak equivalence

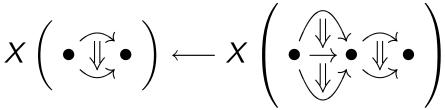
$$X(O_{k-1}) \xrightarrow{\sim} X(O_k)_{\mathrm{inv}} \subseteq X(O_k)$$

for $k = 1, \ldots, n$.

"Composition":
Morphism in Θ_n



induces map of spaces



The wreath category ΘC

C = small category $\Longrightarrow$ category $\Theta C = \Delta \wr C$ (C. Berger):

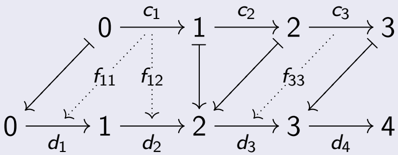
Objects of ΘC

Graphs like

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} 2 \xrightarrow{c_3} 3$$

where $c_i \in \mathrm{ob}C$ (denoted $[3](c_1, c_2, c_3)$).

Morphisms of ΘC



consists of $\delta \colon [3] \to [4] \in \Delta$, $f_{ij} \colon c_i \to d_j \in C$.

Think of $[m](c_1, \ldots, c_m)$ as a C-enriched category

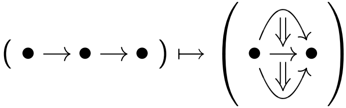
Definition of Θ_n

Definition of Θ_n

$$\Theta_0 \stackrel{\text{def}}{=} 1, \qquad \Theta_n \stackrel{\text{def}}{=} \Theta(\Theta_{n-1})$$

- inclusions $\Theta_1 \subset \Theta_2 \subset \cdots \subset \Theta_n$
- “suspension” $\Theta_{n-1} \to \Theta_n$

$$\theta \mapsto [1](\theta)$$



Definition of $\mathrm{MAP}_X(a,b)$

$X \in \mathrm{Psh}(\Theta_n, \mathrm{Sp})$, $a, b \in X([0]) \implies \mathrm{MAP}_X(a,b) \in \mathrm{Psh}(\Theta_{n-1}, \mathrm{Sp})$:

$$\mathrm{MAP}_X(a,b)(\theta) \stackrel{\text{def}}{=} \text{hofiber}_{(a,b)} \Big[X([1](\theta)) \to X([0]) \times X([0]) \Big]$$

A **Θ_n-space** is a functor $\Theta_n^{\mathrm{op}} \to \mathrm{Sp}$ satisfying the following.

- **Segal condition.** For all $k \geq 2$, $\theta_1, \ldots, \theta_k \in \mathrm{ob}\Theta_{n-1}$,

$$X([k](\theta_1, \ldots, \theta_k)) \xrightarrow{\sim} \lim \left(\begin{array}{ccccccccc} X([1](\theta_1)) & & X([1](\theta_2)) & & \cdots & & X([1](\theta_k)) \\ & \searrow \quad X[0] \quad \swarrow & & \searrow \quad \cdots \quad \swarrow & & \searrow \quad X[0] \quad \swarrow & \end{array} \right)$$

- **Completeness condition.** $X|_{\Theta_1}$ is a complete Segal space
- **Recursive condition.** $\mathrm{MAP}_X(a, b)$ is a Θ_{n-1}-space for all $a, b \in X([0])$
- A Θ_0-space is a space.

Θ_n-spaces as (∞, n)-categories

- Idea: Θ_n-spaces model (∞, n)-categories.
- Θ_n-spaces are local objects for a presentation $(\Theta_n, \mathcal{T})$.
- Not the only model given by a presentation:
  n-**fold complete Segal spaces**, given by a presentation $(\Delta^n, \mathcal{T}')$.
  (Barwick, Lurie).
- These two presentations are different,
  but model the same underlying theory.
  (Underlying model categories are Quillen equivalent.)
- There are other models, not given by a presentation,
  e.g., *n*-**fold Segal categories** (Hirschowitz–Simpson).

$(\Theta_n, \mathcal{T}) \stackrel{\text{def}}{=}$ presentation for Θ_n-spaces.

Theorem (R.)

$(\Theta_n, \mathcal{T})$ is a Cartesian presentation.
$\Longrightarrow$ *if $X, Y \in \mathrm{Psh}(\Theta_n, \mathrm{Sp})$, and Y is a Θ_n-space, so is $\{X, Y\}$.*

- The presentation $(\Delta^n, \mathcal{T}')$ for n-fold complete Segal spaces is not Cartesian (though it comes close).
- The n-fold Segal category model (Hirschowitz–Simpson) gives a Cartesian model category, but isn't given by a presentation.

A more general construction

Let $(C, \mathcal{S})$ be a Cartesian presentation. (Assume C has a terminal object.) There exists a presentation $(\Theta C, \mathcal{T})$, whose local objects X satisfy:

- **Segal condition.** For all $k \geq 2$, $c_1, \ldots, c_k \in \operatorname{ob} C$,

$$X([k](c_1, \ldots, c_k)) \xrightarrow{\sim} \lim \left(\begin{array}{ccccccc} X([1](c_1)) & & X([1](c_2)) & & \cdots & & X([1](c_k)) \\ & \searrow \; X[0] \; \swarrow & & \searrow \; \cdots \; \swarrow & & \searrow \; X[0] \; \swarrow & \end{array} \right)$$

- **Completeness condition.** $X|_{\Theta_1}$ is a complete Segal space
- **Recursive condition.**
  $\mathrm{MAP}_X(a, b) \in \mathrm{Psh}(C, \mathrm{Sp})$ is an $\mathcal{S}$-model for all $a, b \in X([0])$

Theorem

$(\Theta C, \mathcal{T})$ is a Cartesian presentation if $(C, \mathcal{S})$ is

- If $V = \mathrm{Psh}(C, \mathrm{Sp})_{\mathcal{S}}$, then

$$V\text{-}\Theta\mathrm{Sp} \stackrel{\mathrm{def}}{=} \mathrm{Psh}(\Theta C, \mathrm{Sp})_{\mathcal{T}}$$

  should model "$(V, \times)$-enriched categories"
- Theorem says: V Cartesian $\Longrightarrow$ V-$\Theta\mathrm{Sp}$ Cartesian.
- Θ_n-spaces are obtained by iterating the $V \mapsto V$-$\Theta\mathrm{Sp}$ construction, starting with $V = \mathrm{Sp}$

The end

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