WHAT ARE RECOLLEMENTS IN STABLE HOMOTOPY THEORY?

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I keep forgetting how recollements are supposed to work, so I wrote up this note for myself. This is based on Barwick and Glasman, “A note on stable recollements” [BG16], although I have modified the notation somewhat.

You can think of a recollement (in the context of stable ∞-categories) as a very nice kind of “short exact sequence” of stable ∞-categories. However, the notion of recollement comes with a lot of structure, which can express itself in lots of ways. It is a very familiar pattern in stable homotopy theory. It is typicially nearby whenever one encounters something called an “fracture square”.

1. Definition of recollement

In the following, I work with stable ∞-categories. (In practice, they will be presentable stable ∞-categories.) By *full subcategory*, I always mean a replete one, i.e., one which is stable under isomorphisms. By a *localization*, I mean any functor which is characterized by the collection of morphisms it inverts (so a localization is allowed to be a left adjoint or a right adjoint, or both, or neither).

A **recollement** is a sequence of functors between stable ∞-categories of the form

$$\mathbf{U} \overset{i_*}{\rightarrowtail} \mathbf{X} \overset{j^*}{\twoheadrightarrow} \mathbf{Z}$$

with the following properties.

(1) The functor i^* is fully faithful, and is both a left and right adjoint.
(2) The functor j^* is a localization, and is both a left and right adjoint.
(3) The sequence is *exact*, in the sense that $j^*X = 0$ iff X is in the essential image of i_*.

In particular, the above diagram can be extended to one of the form

$$\begin{array}{ccccc} & \overset{i^{\#}}{\twoheadleftarrow} & & \overset{j_{\#}}{\leftarrowtail} & \\ \mathbf{U} & \overset{i_*}{\rightarrowtail} & \mathbf{X} & \overset{j^*}{\twoheadrightarrow} & \mathbf{Z} \\ & \overset{i^!}{\twoheadleftarrow} & & \overset{j_!}{\leftarrowtail} & \end{array}$$

in which $i^{\#}$ and $j_{\#}$ are left adjoints and $i^!$ and $j_!$ are right adjoints. Furthermore, we may identify three full subcategories of $\mathbf{X}$ as essential images of these functors:

$$\mathbf{U} = i_*\mathbf{U}, \qquad \mathbf{Z}^{\vee} = j_{\#}\mathbf{Z}, \qquad \mathbf{Z}^{\wedge} = j_!\mathbf{Z}.$$

Note that typically $\mathbf{Z}^{\vee} \neq \mathbf{Z}^{\wedge}$.

1.1. *Remark.* I’ve chosen the notation so that i-functors always involve $\mathbf{U}$, while j-functors always involve $\mathbf{Z}$. Also, lower index functors $(i_*, j_{\#}, j_!)$ are always fully faithful, while upper index functors $(i^{\#}, i^!, j^*)$ are always localizations. Finally, functors decorated with $\#$ are left adjoints, functors decorated with ! are right adjoints, while functors decorated with $*$ are both.

Given this structure, there are a large number of statements which follow, which can be largely summarized as:

- all three “horizontal sequnces” in the above diagram are exact,
- there are “orthogonality relations” $\mathbf{Z}^{\vee} \perp \mathbf{U} \perp \mathbf{Z}^{\wedge}$ between the various subcategories,

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- there are cofiber sequnces relating the various localizations functors, which
- fit into "fracture squares".

1.2. **Proposition.** *Given a recollement as above, we have the following properties.*

(1) *The subcategory* $\mathbf{U}$ *is a* right orthogonal complement *of* $\mathbf{Z}^\vee$. *That is,* $X \in \mathbf{U}$ *iff* $\mathrm{Hom}(M, X) = 0$ *for all* $M \in \mathbf{Z}^\vee$.

(1') *The subcategory* $\mathbf{U}$ *is a* left orthogonal complement *of* $\mathbf{Z}^\wedge$. *That is,* $X \in \mathbf{U}$ *iff* $\mathrm{Hom}(X, N) = 0$ *for all* $N \in \mathbf{Z}^\wedge$.

(2) *The subcategory* $\mathbf{Z}^\vee$ *is a left orthogonal complement of* $\mathbf{U}$. *That is,* $X \in \mathbf{Z}^\vee$ *iff* $\mathrm{Hom}(X, U) = 0$ *for all* $U \in \mathbf{U}$.

(2') *The subcategory* $\mathbf{Z}^\wedge$ *is a right orthogonal complement of* $\mathbf{U}$. *That is,* $X \in \mathbf{Z}^\wedge$ *iff* $\mathrm{Hom}(U, X) = 0$ *for all* $U \in \mathbf{U}$.

(3) *The sequence of functors* $\mathbf{Z} \xrightarrow{j_\#} \mathbf{X} \xrightarrow{i^\#} \mathbf{U}$ *is exact. That is,* $X \in \mathbf{Z}^\vee$ *iff* $i^\# X = 0$.

(3') *The sequence of functors* $\mathbf{Z} \xrightarrow{j_!} \mathbf{X} \xrightarrow{i^!} \mathbf{U}$ *is exact. That is,* $X \in \mathbf{Z}^\wedge$ *iff* $i^! X = 0$.

(4) *There is a natural cofiber sequence* $j_\# j^* X \xrightarrow{\epsilon} X \xrightarrow{\eta} i_* i^\# X$, *where* η *and* ϵ *are unit and counit of the relevant adjunctions.*

(4') *There is a natural cofiber sequence* $i_* i^! X \xrightarrow{\epsilon} X \xrightarrow{\eta} j_! j^* X$, *where* η *and* ϵ *are unit and counit of the relevant adjunctions.*

(5) *There is a natural pullback square*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & j_! j^* X \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta j_! j^*} \\ i_* i^\# X & \xrightarrow[i_* i^\# \eta]{} & i_* i^\# j_! j^* X \end{array}$$

where η *and* ϵ *are unit and counit of the relevant adjunctions.*

(5') *There is a natural pullback square*

$$\begin{array}{ccc} i_* i^! j_\# j^* X & \xrightarrow{\epsilon j_\# j^*} & j_\# j^* X \\ \downarrow{\scriptstyle i_* i^* \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ i_* i^! X & \xrightarrow[\epsilon]{} & X \end{array}$$

where η *and* ϵ *are unit and counit of the relevant adjunctions.*

Proof. I'll prove (1), (2), (3), (4), and (5), as the other statements are analogous. To prove (1), we use $\mathrm{Hom}_{\mathbf{X}}(j_\# M, X) = \mathrm{Hom}_{\mathbf{Z}}(M, j^* X)$, so that X has no maps from objects of $\mathbf{Z}^\vee$ iff $j^* X = 0$, which holds iff $X \in \mathbf{U}$.

Next I claim that for all $X \in \mathbf{X}$ the cofiber Y of the counit map $\epsilon\colon j_\# j^* X \to X$ is in $\mathbf{U}$. To see this, note that by properties of the adjunction, the composite of

$$j^* X \xrightarrow{\eta j^*} j^* j_\# j^* X \xrightarrow{j^* \epsilon} j^* X$$

is the identity map, where $\eta\colon N \to j^* j_\# N$ is the counit of the adjunction. But since $j_\#$ is fully fathiful, we have that η is an isomorphism whenever $N = j^* X$. Thus $j^* \epsilon$ is an equivalence, whence $j^* Y = 0$, so $Y \in \mathbf{U}$ as desired.

To prove (2), note that we have already shown that $\mathrm{Hom}(M, U) = 0$ for all $M \in \mathbf{Z}^\vee$ and $U \in \mathbf{U}$. Now suppose X is such that $\mathrm{Hom}(X, U) = 0$ for all $U \in \mathbf{U}$. Consider the cofiber sequence $j_\# j^* X \to X \to Y$, and note that we also have $\mathrm{Hom}_{\mathbf{X}}(j_\# j^* X, U) = \mathrm{Hom}_{\mathbf{Z}}(j^* X, j_! U) = 0$ for all $U \in \mathbf{U}$, whence $\mathrm{Hom}(Y, U) = 0$ for all $U \in \mathbf{U}$. But we know $Y \in \mathbf{U}$, so we conclude that $Y = 0$, so $j_\# j^* X \to X$ is an isomorphism, and thus $X \in \mathbf{Z}^\vee$ as desired.

To prove (3), we use $\mathrm{Hom}_{\mathbf{U}}(i^\# X, U) = \mathrm{Hom}_{\mathbf{X}}(X, i_* U)$ and (2) to conclude that X is in the essential image of $j_\#$ iff $i^\# X = 0$.

To prove (4) consider the diagram

$$\begin{array}{ccccc} j_\# j^* X & \xrightarrow{\epsilon} & X & \longrightarrow & Y \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\eta}\downarrow & & {\scriptstyle\eta}\downarrow{\scriptstyle\approx} \\ i_* i^\# j_\# j^* X & \xrightarrow[i_* i^\# \epsilon]{} & i_* i^\# X & \longrightarrow & i_* i^\# Y \end{array}$$

where the top row is a cofiber sequence, and the bottom row is obtained by applying $i_* i^\#$ to the top row, and so is also a cofiber sequence. We have already observed that $Y \in \mathbf{U}$, and therfore $\eta \colon Y \to i_* i^\# Y$ is an isomorphism. Finally, $i^\# j_\# = 0$ by (3) we have that $i_* i^\# j_\# j^* X = 0$. Thus $i_* i^\# X \to Y$ is an isomorphism, exhibiting the desired cofiber sequence.

To prove (5), use (4) and (4') to extend the commutative square to one of the form

$$\begin{array}{ccccc} j_\# j^* i_* i^! X & \xrightarrow{j_\# j^* \epsilon} & j_\# j^* X & \xrightarrow{j_\# j^* \eta} & j_\# j^* j_! j^* X \\ {\scriptstyle \epsilon i_* i^!}\downarrow & & {\scriptstyle\epsilon}\downarrow & & {\scriptstyle \epsilon j_! j^*}\downarrow \\ i_* i^! X & \xrightarrow{\epsilon} & X & \xrightarrow{\eta} & j_! j^* X \\ {\scriptstyle \eta i_* i^!}\downarrow & & {\scriptstyle\eta}\downarrow & & {\scriptstyle \eta j_! j^*}\downarrow \\ i_* i^\# i_* i^! X & \xrightarrow{i_* i^\# \epsilon} & i_* i^\# X & \xrightarrow{i_* i^\# \eta} & i_* i^\# j_! j^* X \end{array}$$

in which all rows and columns are cofiber sequences. The claim follows since $j^* i_* = 0$, so the upper-left corner is 0. □

We can explicitly reconstruct $\mathbf{X}$ using data based on the subcategories $\mathbf{U}$ and $\mathbf{Z}^\wedge$, or $\mathbf{U}$ and $\mathbf{Z}^\vee$.

1.3. **Proposition.** *The functor*

$$\mathbf{Z}^\wedge \times_{\mathbf{X}} \mathrm{Fun}(\Delta^1, \mathbf{X}) \times_{\mathbf{X}} \mathbf{U} \to \mathbf{X}$$

which sends a morphism $Z \to U$ to its fiber is an equivalence. The functor

$$\mathbf{U} \times_{\mathbf{X}} \mathrm{Fun}(\Delta^1, \mathbf{X}) \times_{\mathbf{X}} \mathbf{Z}^\vee \to \mathbf{X}$$

which sends a morphism $U \to Z$ to its fiber is an equivalence.

Proof idea. (See Barwick-Glasman, Lemma 9.) By property (4') above, any $X \in \mathbf{X}$ fits in a fiber sequence $X \xrightarrow{\eta} j_! j^* X \to \Sigma i_* i^! X$ with $j_! j^* X \in \mathbf{Z}^\wedge$ and $i_* i^! X \in \mathbf{U}$. Conversely, given any fiber sequence $X \to Z \to U$ with $Z \in \mathbf{Z}^\wedge$ and $U \in \mathbf{U}$, we have $j_! j^* X \xrightarrow{\sim} j_! j^* Z \approx Z$ since $j^* U = 0$ and $U \approx i_* i^! U \xrightarrow{\sim} \Sigma i_* i^! X$ since $i^! Z = 0$.

Similarly by property (4) above, any $X \in \mathbf{X}$ fits in a fiber sequence $X \xrightarrow{\epsilon} i_* i^\# X \to \Sigma j_\# j^* X$ with $i_* i^\# X \in \mathbf{U}$ and $j_\# j^* X \in \mathbf{Z}^\vee$. Conversely, given any fiber sequence $X \to U \to Z$ with $U \in \mathbf{U}$ and $Z \in \mathbf{Z}^\vee$, we have $i_* i^\# X \xrightarrow{\sim} i_* i^\# U \approx U$ since $i^\# Z = 0$ and $Z \approx j_\# j^* Z \xrightarrow{\sim} \Sigma j_\# j^* X$ since $j^* U = 0$. □

2. The symmetric monoidal case

Let's think about what this looks like when $\mathbf{U}, \mathbf{X}, \mathbf{Z}$ are presentably symmetric monoidal, and the localization functors $i^\# \colon \mathbf{X} \to \mathbf{U}$ and $j^* \colon \mathbf{X} \to \mathbf{Z}$ are strongly symmetric monoidal. Then we can describe the situation as follows.

- Let $L := i_* i^\# \mathbb{1}$. Then, with respect to the unit map $\mathbb{1} \to L$ of the adjunction, L is an idempotent ring.

- Let $C := j_\# j^* \mathbb{1}$. Then, with respect to the counit map $C \to \mathbb{1}$ of the adjunction, C is an idempotent coring.
- These counit and unit maps fit into a cofiber sequence $C \xrightarrow{\epsilon} \mathbb{1} \xrightarrow{\eta} L$.
- We have $\mathbf{U} = \mathrm{Mod}(L)$ and $\mathbf{Z}^\vee = \mathrm{Comod}(C)$, with the inclusions into $\mathbf{X}$ corresponding to the evident forgetful functors.
- The localization monads and comonads are described by

$$\begin{array}{ccc} i_* i^\# = L\otimes - & & j_\# j^* = C \otimes - \\ & \circlearrowright \mathbf{X} \circlearrowleft & \\ i_* i^! = [\![L , -]\!] & & j_! j^* = [\![C , -]\!] \end{array}$$

so that the cofiber sequences relating them have the form

$$C \otimes X \to \mathbb{1} \otimes X \to L \otimes X, \qquad [\![L , X]\!] \to [\![\mathbb{1} , X]\!] \to [\![C , X]\!].$$

Here I write $[\![X , Y]\!]$ for an internal function object.
- The arithmetic squares have the form

$$\begin{array}{ccc} X & \longrightarrow & [\![C , X]\!] \\ \downarrow & & \downarrow \\ L \otimes X & \longrightarrow & L \otimes [\![C , X]\!] \end{array} \qquad \begin{array}{ccc} [\![L , C \otimes X]\!] & \longrightarrow & C \otimes X \\ \downarrow & & \downarrow \\ [\![L , X]\!] & \longrightarrow & X \end{array}$$

3. Some examples

Here are some standard examples of recollements which appear in stable homotopy theory. Because full subcategories seem easier to visusualize than localizations, I'll often exhibit these diagramatically in the form:

$$\begin{array}{ccccc} & \overset{i^\#}{\twoheadleftarrow} & & \longleftarrow\!\!\!\prec & \mathbf{Z}^\vee \\ \mathbf{U} & \rightarrowtail & \mathbf{X} & \overset{j^*}{\underset{j^*}{\twoheadrightarrow}} & \wr\wr \\ & \underset{i^!}{\twoheadleftarrow} & & \longleftarrow\!\!\!\prec & \mathbf{Z}^\wedge \end{array}$$

In this case, the functor $j^* \colon \mathbf{X} \to \mathbf{Z}$ may have two different descriptions, according to which subcategory is identified with its codomain.

3.1. *Example* (p-completion). This is:

$$\begin{array}{ccccc} & \overset{S[1/p]\otimes -}{\twoheadleftarrow} & & \longleftarrow\!\!\!\prec & (\mathrm{Mod}_{\mathbb{S}})^{p\text{-tors}} \\ \mathrm{Mod}_{\mathbb{S}[1/p]} & \rightarrowtail & \mathrm{Mod}_{\mathbb{S}} & \overset{\Sigma^{-1}\mathbb{S}/p^\infty \otimes -}{\underset{[\![\Sigma^{-1}\mathbb{S}/p^\infty , -]\!]}{\twoheadrightarrow}} & \wr\wr \\ & \underset{[\![\mathbb{S}[1/p] , -]\!]}{\twoheadleftarrow} & & \longleftarrow\!\!\!\prec & (\mathrm{Mod}_{\mathbb{S}})^\wedge_p \end{array}$$

where $\mathbf{X} = \mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$ is spectra, $\mathbf{U} = \mathrm{Mod}_{\mathbb{S}[1/p]}$ is the full subcategory of spectra on which p is an isomorphism, $\mathbf{Z}^\wedge = (\mathrm{Mod}_{\mathbb{S}})^\wedge_p$ is the full subcategory of p-complete spectra, and $\mathbf{Z}^\vee = (\mathrm{Mod}_{\mathbb{S}})^{p\text{-tors}}$ is the full subcategory of p-torsion spectra. Then $L = \mathbb{S}[1/p]$ and $C = \Sigma^{-1}\mathbb{S}/p^\infty$. The arithmetic square is

$$\begin{array}{ccc} X & \longrightarrow & X_p \\ \downarrow & & \downarrow \\ \mathbb{S}[1/p] \otimes X & \longrightarrow & \mathbb{S}[1/p] \otimes X_p \end{array}$$

where $X_p = [\![\Sigma^{-1}\mathbb{S}/p^\infty , X]\!]$ is the p-completion of X.

3.2. *Example* (L_n^f localization). This is:

$$
\begin{array}{ccccc}
 & \xleftarrow{\;L_n^f\mathbb{S}\otimes-\;} & & \longleftarrow & C_n^f\mathrm{Sp} \\
L_n^f\mathrm{Sp} & \rightarrowtail & \mathrm{Sp} & \underset{[\![C_n^f\mathbb{S},-]\!]}{\overset{C_n^f\mathbb{S}\otimes-}{\twoheadrightarrow}} & \wr\!\wr \\
 & \xleftarrow[\;[\![L_n^f\mathbb{S},-]\!]\;]{} & & \longleftarrow & \mathrm{Sp}_{C_n^f\mathbb{S}}
\end{array}
$$

where $\mathbf{X} = \mathrm{Sp}$ is the category of spectra, $\mathbf{U} = L_n^f\mathrm{Sp}$ is the full subcategory of L_n^f-local spectra, $\mathbf{Z}^\wedge = \mathrm{Sp}_{C_n^f\mathbb{S}}$ is the full subcategory of $C_n^f\mathbb{S}$-local spectra, and $\mathbf{Z}^\vee = C_n^f\mathrm{Sp}$ is the full subcategory spanned by X such that $L_n^f X = 0$. (Here $C_n^f X =$ fiber of $X \to L_n^f X$.) Then $L = L_n^f\mathbb{S}$ and $C = C_n^f\mathbb{S}$. The arithmetic square is

$$
\begin{array}{ccc}
X & \longrightarrow & [\![C_n^f\mathbb{S}, X]\!] \\
\big\downarrow & & \big\downarrow \\
L_n^f X & \longrightarrow & L_n^f[\![C_n^f\mathbb{S}, X]\!]
\end{array}
$$

3.3. *Example* (Monochromatic localization). This is:

$$
\begin{array}{ccccc}
 & \xleftarrow{\;L_{n-1}^f\mathbb{S}\otimes-\;} & & \longleftarrow & M_n^f\mathrm{Sp} \\
L_{n-1}^f\mathrm{Sp} & \rightarrowtail & L_n^f\mathrm{Sp} & \underset{L_{T(n)}}{\overset{M_n^f}{\twoheadrightarrow}} & \wr\!\wr \\
 & \xleftarrow[\;[\![L_{n-1}^f\mathbb{S},-]\!]\;]{} & & \longleftarrow & \mathrm{Sp}_{T(n)}
\end{array}
$$

where $\mathbf{X} = L_n^f\mathrm{Sp}$ is the full subcategory of L_n^f-local spectra, $\mathbf{U} = L_{n-1}^f\mathrm{Sp}$ is the full subcategory of L_{n-1}^f-local spectra, $\mathbf{Z}^\vee = M_n^f\mathrm{Sp}$ is the full subcategory of n-monomochromatic spectra (i.e., L_n^f-local and killed by L_{n-1}^f), and $\mathbf{Z}^\wedge = \mathrm{Sp}_{T(n)}$ is the full subcategory of $T(n)$-local spectra, where $T(n)$ is a telescope on a type n finite spectrum. Then $L = L_{n-1}^f\mathbb{S}$ and $C = L_n^f C_{n-1}^f\mathbb{S}$. The arithmetic square is

$$
\begin{array}{ccc}
L_n^f X & \longrightarrow & L_{T(n)}X \\
\big\downarrow & & \big\downarrow \\
L_{n-1}^f X & \longrightarrow & L_{n-1}^f L_{T(n)}X
\end{array}
$$

3.4. *Example* (Equavariant spectra and isotropy separation). Recall that unstable G-equivariant homotopy theory $\mathcal{S}_G$ is equivalent to $\mathrm{PSh}(\mathrm{Orb}_G)$, presheaves on the orbit category of G. Note that in this category the diagonal map of EG is an equivalence: $EG \xrightarrow{\sim} EG \times EG$. The cofiber of the projection $EG \to *$ is denoted $\widetilde{EG}$.

Let $\mathbf{X} = \mathrm{Sp}_G$ be the category of genuine G-spectra. Let $C = \Sigma_+^\infty EG$ and $L = \Sigma^\infty \widetilde{EG}$. Then we get a recollement:

$$
\begin{array}{ccccc}
 & \xleftarrow{\;\Sigma^\infty\widetilde{EG}\otimes-\;} & & \longleftarrow & \mathrm{Sp}_G^{\mathrm{free}} \\
\mathrm{Sp}_G^{\text{u-acyc}} & \rightarrowtail & \mathrm{Sp}_G & \underset{[\![\Sigma_+^\infty EG,-]\!]}{\overset{\Sigma_+^\infty EG\otimes-}{\twoheadrightarrow}} & \wr\!\wr \\
 & \xleftarrow[\;[\![\Sigma^\infty\widetilde{EG},-]\!]\;]{} & & \longleftarrow & \mathrm{Sp}_G^{\mathrm{Borel}}
\end{array}
$$

where $\mathbf{Z}^\vee = \mathrm{Sp}_G^{\mathrm{free}}$ is the full subcategory of free G-spectra, $\mathbf{Z}^\wedge = \mathrm{Sp}_G^{\mathrm{Borel}}$ is the full subcategory of Borel G-spectra, and $\mathbf{U} = \mathrm{Sp}_G^{\text{u-acyc}}$ is the full subcategory of G-spectra whose underlying ordinary spectrum is contractible.

We can also identify $\mathbf{Z} = \mathrm{Sp}^{BG} = \mathrm{Mod}_{\mathbb{S}[G]}$, the category of spectra equipped with a G-action, in which $j^*\colon \mathrm{Sp}_G \to \mathrm{Sp}^{BG}$ the underlying spectrum funtor, i.e., $[\Sigma^\infty_+ G/e, -]$, the functor $j_\#\colon \mathrm{Sp}^{BG} \to \mathrm{Sp}_G$ is given by $\Sigma^\infty_+ G/e \otimes_{\mathbb{S}[G]} -$, and $j_!\colon \mathrm{Sp}^{BG} \to \mathrm{Sp}_G$ is “Borelification”.

The associated arithmetic square is

$$\begin{array}{ccc} X & \longrightarrow & [\Sigma^\infty_+ EG, X] \\ \downarrow & & \downarrow \\ \Sigma^\infty \widetilde{EG} \otimes X & \longrightarrow & \Sigma^\infty \widetilde{EG} \otimes [\Sigma^\infty_+ EG, X] \end{array}$$

If $G = C_p$ is a prime cyclic group, then taking G-fixed points of this square (i.e., the mapping spectrum from $\mathbb{S}$) gives a pullback square in Sp of the form

$$\begin{array}{ccc} X^G & \longrightarrow & X^{hG} \\ \downarrow & & \downarrow \\ \Phi^G X & \longrightarrow & X^{tG} \end{array}$$

where X^{hG} is the G-fixed points of the underlying spectrum with G-action, Φ^G is geometric fixed points, and X^{tG} is the Tate spectrum. (Much of this can be found in Dennis Nardin, “Introduction to equivariant homotopy theory” [Nar20]. See also Mathew-Naumann-Noel, “Nilpotence and descent in equivariant stable homotopy theory” [MNN17].) These ideas extend more general isotropy separation diagrams associated to more general groups, though I'm not sure how to state it cleanly. (See Glasman, “Stratified categories, geometric fixed points, and a generalized Arone-Ching theorem” [Gla15].)

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