Isogenies, power operations, and homotopy theory

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Abstract. The modern understanding of the homotopy theory of spaces and spectra is organized by the chromatic philosophy, which relates phenomena in homotopy theory with the moduli of one-dimensional formal groups. In this paper, we describe how certain phenomena in $K(n)$-local homotopy can be computed from knowledge of isogenies of deformations of formal groups of height n.

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1. Introduction

A sweeping theme in the study of homotopy theory over the past several decades is the *chromatic viewpoint.* In this philosophy, phenomena in homotopy theory are associated to phenomena in the theory of one-dimensional formal groups. This program was instigated by Quillen's observation of the connection between complex bordism and formal group laws [Qui69].

The chromatic picture is best described in terms of localization at a chosen prime p. After one localizes at a prime p, the moduli of formal groups admits a descending filtration, called the *height* filtration. According the chromatic philosophy, this filtration is mirrored by a sequence of successive approximations to homotopy theory. The difference between adjacent approximation is the *nth chromatic layer*, which is associated by the chromatic picture to formal groups of height n. Phenomena in the nth chromatic layer may be detected using cohomology theories called *Morava K-theories* and *Morava E-theories*, which are typically (and unimaginatively) denoted $K(n)$ and E_n. A good recent introduction to this point of view is is [Goe].

In this paper I will describe a particular manifestation of the chromatic picture, which relates "$K(n)$-local homotopy theory" (i.e., one manifestation of the nth chromatic layer in homotopy theory), to *isogenies* of formal groups.

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2. Formal groups and localized homotopy theory

We briefly recall the role and significance of formal groups in homotopy theory.

2.1. Complex orientation and formal groups. Recall that a generalized cohomology theory E^* is said to be **complex orientable** if (i) it takes values in graded rings, and (ii) if there exists an element $x \in \widetilde{E}^*\mathbb{CP}^\infty$ which restricts to the fundamental class in $\widetilde{E}^*\mathbb{CP}^1$. To such a theory is associated a (one-dimensional, commutative) **formal group** $\mathbb{G}_E$, which is the formal scheme over the ring E^* with coefficient ring $\mathcal{O}_{\mathbb{G}_E} = E^*\mathbb{CP}^\infty$.

Remark 2.1. For a complex orientable theory, a choice of element x as in (ii) gives rise to an "Euler class" on complex line bundles, defined by $e_x(L \to X) := f^*(x) \in E^*(X)$ where $f\colon X \to \mathbb{CP}^\infty$ classifies L, together with a power series $F_x(t_1, t_2) \in E^*[\![t_1, t_2]\!]$ expressing the Euler class of a tensor product of lines:

$$e_x(L_1 \otimes L_2) = F_x(e_x(L_1), e_x(L_2)),$$

which is an example of a **formal group law** on E^*. Both e_x and F_x depend on the choice of "coordinate" $x \in \mathcal{O}_{\mathbb{G}_E} = E^*\mathbb{CP}^\infty$. The formal group $\mathbb{G}_E$ is a coordinate-free expression of the collection of formal group laws associated to E, and depends only on the cohomology theory E itself.

Example 2.1. For a ring R, let HR^* denote ordinary cohomology with coefficients in R. For any R, the theory HR^* is complex orientable, and the resulting formal group $\mathbb{G}_R$ is the *additive* formal group. In fact, if we take $x \in HR^2(\mathbb{CP}^\infty)$ to be any generator, we have that $F_x(t_1, t_2) = t_1 + t_2$, and recover the classical addition formula for first Chern classes of complex line bundles.

Example 2.2. Complex bordism MU^* is a complex oriented theory, which comes with a tautological choice of coordinate $x \in MU^2\mathbb{CP}^\infty$. Quillen [Qui69] identified the resulting formal group F_x as the *universal* formal group law. In coordinate-free language, we may say that the formal group $\mathbb{G}_{MU}$ of complex bordism is the universal example of a formal group equipped with the data of a choice of coordinate.

All formal groups over a given field of characteristic 0 are isomorphic to the additive formal group. For a formal groups $\mathbb{G}$ over fields k of characteristic p, there is an isomorphism invariant called the **height** of $\mathbb{G}$, which is an element $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. For separably closed k, the height is a complete invariant.

Example 2.3. Fix a prime p. For any height $n \in \mathbb{Z}_{\geq 1}$, there exist complex cohomology theories whose formal group is one of height n. The standard examples are the **Morava K-theories** $K(n)$, whose coefficient ring is $K(n)^* = \mathbb{F}_p[v_n^{\pm}]$ with $v_n \in K(n)^{-2(p^n-1)}$, and whose formal group is the **Honda formal group** of height n.

Example 2.4. For any formal group G_0 of height $1 \leq n < \infty$ over a perfect field k, Lubin and Tate constructed its *universal deformation*, which is a formal group G defined over the complete local ring $\mathbb{W}_p k[\![u_1, \dots, u_{n-1}]\!]$, whose restriction at the special fiber is G_0. There is a corresponding cohomology theory called **Morava E-theory**, which will play an important role in our story; see §4.2 below.

Formal groups of infinite height over fields of characteristic p are isomorphic to the additive formal group. Ordinary homology $H\mathbb{F}_p$ with mod p coefficients is an example of a complex oriented theory whose formal group has infinite height.

It is conventional to say that any formal group over a field of characteristic zero has height 0. Ordinary homology $H\mathbb{Q}$ with rational coefficients is an example of a theory with such a formal group.

2.2. Localized homotopy theory.

Associated to any homology theory E is a corresponding **localization functor**, first constructed in full generality by Bousfield [Bou75]. Say that a based space Y is **E-local** if for any based CW-complex K such that $\widetilde{E}_*(K) \approx 0$, the space $\mathrm{Map}_*(K, Y)$ of based maps is weakly contractible. Bousfield showed that for any space X, there exists a space X_E and a map $\eta_E \colon X \to X_E$, called the **E-localization** of X, such that (i) the map η_E induces an isomorphism in E-homology, and (ii) X_E is E-local. Furthermore, the operation $X \mapsto X_E$ can be realized functorially.

Example 2.5. For ordinary homology theories $E = HR$, localization of spaces is well-behaved in the absence of fundamental groups. For instance, $\pi_*(X_{H\mathbb{Q}}) \approx \pi_* X \otimes \mathbb{Q}$ if X is simply connected, and $\pi_*(X_{H\mathbb{F}_p}) \approx (\pi_* X)_p^\wedge$ if X is simply connected and finite type.

There is an analogous localization construction for spectra. In what follows we will be most interested in localization with respect to Morava K-theory spectra. In particular, for every prime p and $n \geq 1$, there is a localization functor

$$X \mapsto X_{K(n)} \colon h\mathrm{Spectra} \to h\mathrm{Spectra}_{K(n)} \subset h\mathrm{Spectra}$$

from the homotopy category of spectra to the full subcategory of $K(n)$-local spectra.

2.3. The functor of Bousfield and Kuhn.

It is a remarkable observation of Bousfield [Bou87] and Kuhn [Kuh89] that localization functors on spectra with respect to certain homology theories (such as Morava K-theories) actually factor through the underlying space.

Fix a prime p and an integer $n \geq 1$. There exists a functor

$$\Phi_n \colon \mathrm{Spaces}_* \to \mathrm{Spectra}_{K(n)} \subset \mathrm{Spectra}$$

from pointed spaces to $K(n)$-local spectra which makes the following diagram commute up to natural weak equivalence.

$$\begin{tikzcd}
\mathrm{Spectra} \arrow[rr, "(-)_{K(n)}"] \arrow[dr, "\Omega^\infty"'] & & \mathrm{Spectra}_{K(n)} \\
& \mathrm{Spaces}_* \arrow[ur, "\Phi_n"'] &
\end{tikzcd}$$

The functor Φ_n is constructed using the existence of periodic phenomena in stable homotopy theory [HS98]. Observe that given any space Y and map $g\colon Y \to \Omega^d Y$ with $d \geq 1$, we can obtain a spectrum E by setting $\underline{E}_{kd} = Y$ using the g as the structure map $\underline{E}_{kd} \to \Omega^d \underline{E}_{kd+d}$ (much as periodic K-theory is obtained from the Bott periodicity map $U \to \Omega^2 U$, though in our case g need not be an equivalence). Given a based finite CW-complex K and a map $f\colon \Sigma^d K \to K$ for some $d \geq 1$, define a functor $\Phi_{K,f}\colon \mathrm{Spaces}_* \to \mathrm{Spectra}$ by associating to a based space X the map

$$f^*\colon \operatorname{Map}_*(K, X) \xrightarrow{f} \operatorname{Map}_*(\Sigma^d K, X) \approx \Omega^d \operatorname{Map}_*(K, X),$$

from which we obtain a spectrum $\Phi_{K,f}(X)$ as above. The functor $\Phi_{K,f}(X)$ has the properties:

- $\Phi_{K,f}(\Omega X) \approx \Omega \Phi_{K,f}(X)$,
- If $X = \Omega^\infty Y$ is the 0-space of a spectrum Y, then

  $$\Phi_{K,f}(\Omega^\infty Y) \approx \operatorname{hocolim}_k \Sigma^{-kd} \underline{\mathrm{Hom}}(\Sigma^\infty K, Y) \approx (\Sigma^\infty f)^{-1} \underline{\mathrm{Hom}}(\Sigma^\infty K, Y),$$

  the "telescope" of the function spectrum $\underline{\mathrm{Hom}}(\Sigma^\infty K, Y)$ induced by the map $\Sigma^\infty f$.

Non-trivial examples are provided by a v_n-self map, i.e., a pair (K, f) such that K is a finite CW complex with $K(n)_* K \neq 0$ and $f\colon \Sigma^d K \to K$ with $d \geq 1$ such that $K(n)_* f$ is an isomorphism. (This implies that for any spectrum Y, the map $\underline{\mathrm{Hom}}(K, Y) \to f^{-1}\underline{\mathrm{Hom}}(K, Y)$ induces an isomorphism on $K(n)_*$-homology, and thus $\Phi_{K,f}$ is seen to be non-trivial.)

Such v_n-self maps are plentiful by the *periodicity theorem* of Hopkins-Smith [HS98]. Using this theory, [Kuh89] constructs Φ_n^f as a homotopy inverse limit of a suitably chosen family of functors $\Sigma^{-q_i} \Phi_{K_i, f_i}$; then $\Phi_n(X) := \Phi_n^f(X)_{K(n)}$.

Remark 2.2. Given a v_n-self map f of K, the homotopy groups $\pi_* \Phi_{K,f}(X) \approx f^{-1}\pi_* \operatorname{Map}(K, X)$ are the v_n**-periodic homotopy groups** of the space X, usually denoted $v_n^{-1}\pi_*(X; K)$ (they depend on K, but not on the choice of v_n-self map f). As a result, the spectrum $\Phi_n(X)$ contains information about the v_n-periodic homotopy groups of the space X. The extent to which this information is captured depends in part on the status of the *telescope conjecture*, which if true would imply that $\Phi_n = \Phi_n^f$; see the discussion in [Kuh08].

Bousfield has developed a theory to effectively compute invariants of $\Phi_1(X)$ for certain spaces X, such as spheres and finite H-spaces [Bou99], [Bou05]. In §5, we will outline an approach to generalize Bousfield's results to the case of $n \geq 2$.

2.4. The Bousfield-Kuhn idempotent. Given a basepoint preserving unstable map $f\colon \Omega^\infty F \to \Omega^\infty E$ where E is a $K(n)$-local spectrum, the $K(n)$-local Bousfield-Kuhn functor gives rise to a map of spectra

$$F \xrightarrow{\iota} F_{K(n)} \approx \Phi_n \Omega^\infty F \xrightarrow{\Phi_n(f)} \Omega^\infty E \approx E,$$

and hence an infinite loop space map

$$\phi_n(f) := \Omega^\infty(\Phi_n(f)\circ \iota)\colon \Omega^\infty F \to \Omega^\infty E.$$

If $f = \Omega^\infty g$ for a map $g\colon F \to E$ of spectra, then $\Phi_n(f)\circ\iota = g$. Thus, the function

$$\phi_n\colon h\mathrm{Spaces}_*(\Omega^\infty F, \Omega^\infty E) \xrightarrow{\Phi} h\mathrm{Spectra}(F, E) \xrightarrow{\Omega^\infty} h\mathrm{Spaces}_*(\Omega^\infty F, \Omega^\infty E)$$

is *idempotent*, with image precisely the set of homotopy classes of maps $\Omega^\infty F \to \Omega^\infty E$ which are infinite loop maps.

It turns out to be possible to compute something about the map $\phi_n(f)$ (as an *unstable* map), when E is a complex oriented cohomology theory to which the character theory of Hopkins-Kuhn-Ravenel [HKR00] can be applied, such as Morava E-theory. For the purposes of stating a result, we recall that the Hopkins-Kuhn-Ravenel theory provides a natural ring homomorphism

$$\chi\colon E^0(X\times BG) \to \prod_{g\in G_{n,p}} D_N \otimes_{E^0} E^0(X),$$

for any finite group G, where: X is a finite CW-complex, $G_{n,p}$ is the set of conjugacy classes of homomorphisms $\Lambda_N = (\mathbb{Z}/p^N)^n \to G$ (for N sufficiently large, depending on G), and D_N is a certain faithfully flat extension of E^0.

Let $f\colon \Omega^\infty F \to \Omega^\infty E$ be an H-map; i.e., the induced operation $F^0(-) \to E^0(-)$ is a group homomorphism. The following computes the operation $\phi_n(f)$.

Theorem 2.3. *Let E be a $K(n)$-local complex orientable theory such that π_*E is a complete local ring, and let $f\colon \Omega^\infty F \to \Omega^\infty E$ be an H-space map. Then for any finite CW-complex X, the map*

$$\phi_n(f)\colon F^0(X) \to E^0(X)$$

is given (modulo torsion in $E^0(X)$) by

$$(\phi_n f)(x) = \sum_{k=0}^{n} (-1)^k p^{\binom{k}{2}} \left(p^{-k} \sum_{A\subseteq \Lambda_1^*,\ |A|=p^k} \chi(f(\widetilde{x}\wedge t_A))(g_A). \right) \tag{2.1}$$

The inner sum is taken over all subgroups of $\Lambda_1^* = \mathrm{Hom}(\Lambda_1, U(1))$ of given order; the map $g_A\colon \Lambda_1 \to A^* = \mathrm{Hom}(A, U(1))$ is the dual homomorphism to the inclusion $A \subseteq \Lambda_1^*$; $t_A\colon \Sigma_+^\infty BA^* \to S^0$ is the stable transfer map, and $\widetilde{x}\colon \Sigma_+^\infty X \to F$ is the map of spectra representing $x \in F^0(X)$.

It turns out that the inner sum of (2.1) in fact takes values in $E^0(X) \subset D_1 \otimes_{E^0} E^0(X)$, and furthermore this value is divisible by p^k, so that the bracketed expression in (2.1) gives a well-defined element of $E^0(X)$ modulo torsion. The statement (2.3) has not appeared in print elsewhere, but it is a consequence of the methods of [Rez06].

Example 2.6. Let $E = K_p$ be p-complete K-theory. Then the formula of (2.3) becomes

$$(\Phi f)(x) = f(x) - p^{-1}\chi(f(\widetilde{x} \wedge t_{Z/p}))(g_{\mathbb{Z}/p}).$$

In particular, if $f\colon \Omega^\infty F \to \Omega^\infty K_p$ is an H-map such that $\chi(f(\widetilde{x} \wedge t_{\mathbb{Z}/p}))(g_{\mathbb{Z}/p}) = 0$, then f admits the structure of an infinite loop map. This immediately recovers a well-known theorem of Madsen-Snaith-Tornehave [MST77].

There is a generalization of (2.1) for f which is not necessarily an H-map, though it is too cumbersome to give it here. Another approach to producing formulas for ϕ_n, where the target spectrum is $E = K(n)$, is given in [SW08].

3. Units and orientations

The Bousfield-Kuhn idempotent can be usefully applied to the study of the units spectrum of a commutative S-algebra.

3.1. The units of a commutative ring spectrum. Let R be a homotopy associative ring spectrum. The **units space** of R is called $GL_1(R)$; it is defined by the pullback square of spaces

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d] \\ (\pi_0 R)^\times \arrow[r, tail] & \pi_0 R \end{tikzcd}$$

For a space X, we have $h\mathrm{Spaces}(X, GL_1(R)) \approx R^0(X)^\times \subseteq R^0(X)$.

When R is a commutative S-algebra, then $GL_1(R)$ admits a canonical structure of a grouplike E_∞-space, induced by the multiplicative structure of R. Write $\mathrm{gl}_1(R)$ for the (-1)-connected spectrum which is the infinite delooping of $GL_1(R)$, called the **units spectrum** of R.

The units spectrum carries the obstruction to constructing orientations of commutative S-algebras, as shown by May, Quinn, Ray, and Tornehave in [May77]; see [May09] and [ABG$^+$] for recent treatments of this theory. Let $f\colon g \to o$ be a map of (-1)-connective spectra, where o is the infinite delooping of BO, the classifying space of the infinite dimensional orthogonal group. Let $BG = \Omega^\infty g$ denote the infinite delooping of g, and let MG denote the Thom spectrum of the virtual vector bundle classified by $B(\Omega^\infty f)\colon BG \to BO$; the spectrum MG admits (up to weak equivalence) the structure of a commutative S-algebra. Then one can show that space of commutative S-algebra maps $MG \to R$ is weakly equivalent to the space of null-homotopies of the composite map

$$g \xrightarrow{f} o \xrightarrow{j} \mathrm{gl}_1(S) \to \mathrm{gl}_1(R).$$

Thus, understanding the homotopy type of the spectrum $\mathrm{gl}_1(R)$ is essential to understanding G-orientations of R which are realized by maps of commutative S-algebras.

3.2. A "logarithmic" operation. We use the Bousfield-Kuhn functor to obtain information about the $K(n)$-localization of $\mathrm{gl}_1(R)$. To do this, we consider the "shift" map

$$s\colon GL_1(R) \xrightarrow{x\mapsto x-1} \Omega^\infty R.$$

This shift map is a based map between infinite loop spaces, and thus we may apply the idempotent operator of §2.4 to it. If R is a $K(n)$-local commutative S-algebra, we obtain in this way from this cohomology operation of the form

$$\ell_n = \phi_n(s)\colon R^0(X)^\times \to R^0(X),$$

which is "logarithmic", in the sense that $\ell_n(xy) = \ell_n(x) + \ell_n(y)$. The operation ℓ_n is represented by a map of spectra $\mathrm{gl}_1(R) \to R$.

Example 3.1. To get a sense of what such an operation provides, consider the following analogous situation, where E is a *rational* commutative S-algebra. For any pointed and *connected* space X, we can define a group homomorphism

$$\ell_{\mathbb{Q}}\colon (1+\overline{E}^0(X))^\times \to E^0(X) \qquad \text{by} \qquad \ell_{\mathbb{Q}}(x) = -\textstyle\sum_{m\geq 1}(1-x)^m/m = \log(x).$$

The series defining $\ell_{\mathbb{Q}}$ is converges: because X is connected, $1-x$ is nilpotent when restricted to any connected finite CW-complex mapping to X. The operation $\ell_{\mathbb{Q}}$ is in fact stable: it is represented by a map of spectra $(\mathrm{gl}_1 E)_{\geq 1} \to E$ (where $Z_{\geq n}$ denotes the $(n-1)$-connected cover of a spectrum Z).

The above theory applies in this case to give the following.

Theorem 3.1 ([Rez06]). *Let E be a Morava E-theory* (2.4)*, associated to the Lubin-Tate universal deformation of a height n-formal group. Then its logarithmic operation is given (modulo torsion) by the formula*

$$\ell_n(x) = \frac{1}{p}\log\left(\prod_{k=0}^{n}\Bigg(\prod_{A\subset\Lambda^*,\ |A|=p^k}\psi_A(x)\Bigg)^{(-1)^k p^{\binom{k}{2}-k+1}}\right). \tag{3.1}$$

The functions $\psi_A\colon E^0(X) \to D\otimes_{E^0} E^0(X)$ are certain natural additive and multiplicative cohomology operations (described below (4.1)*), and* $\log(x) = -\sum_{m\geq 1}(1-x)^m/m$*.*

The expression inside "log" in (3.1) is a multiplicative analog of (2.1), with the role of $x \mapsto \chi(f(\widetilde{x}) \wedge t_A, g_A)$ in (2.1) replaced by the operation ψ_A. It turns out that the expression inside log in (3.1) is in fact contained in $E^0(X) \subseteq D \otimes_{E^0} E^0(X)$, and is congruent to 1 modulo p; thus evaluating the formal expansion of log at this expression converges p-adically to an element of $E^0(X)$.

Example 3.2 (tom Dieck's logarithm). An example of a Morava E-theory spectrum at height 1 is KU_p, the p-completion of complex K-theory. In fact, it is possible to generalize (3.1) in the case of $n = 1$ to any $K(1)$-local commutative S-algebra E, so we will describe the result in this case [Rez06, Thm. 1.9]. The formula (3.1) takes the form

$$\ell_1(x) = \frac{1}{p} \log\bigl(x^p/\psi^p(x)\bigr) = -\sum_{m \geq 1} \frac{1}{pm}\bigl(1 - x^p/\psi^p(x)\bigr)^m. \tag{3.2}$$

If E is KU_p or KO_p, the operation $\psi^p \colon E^0(-) \to E^0(-)$ is the usual pth Adams operation on p-complete real-or-complex K-theory.

We can do a little better in this case: the operation ψ^p on $E^0(-)$ satisfies a "Frobenius congruence" $\psi^p(x) \equiv x^p \mod pE^0(-)$; therefore the infinite series of (3.2) converges p-adically. The Frobenius congruence is "witnessed" by a cohomology operation $\theta^p \colon E^0(-) \to E^0(-)$, satisfying the identity $\psi^p(x) = x^p + p\,\theta^p(x)$. Thus we can write

$$\ell_1(x) = \sum_{m \geq 1} (-1)^m \frac{p^{m-1}}{m} (\theta^p(x)/x^p)^m, \tag{3.3}$$

and (3.3) in fact holds on the nose (i.e., not merely modulo torsion [Rez06, Thm. 1.9]). The right-hand side of (3.3) recovers the **Artin-Hasse logarithm** of tom Dieck [tD89], who originally realized this operation as as spectrum maps $\mathrm{gl}_1(KU_p) \to KU_p$ and $\mathrm{gl}_1(KO_p) \to KO_p$ without reference to the Bousfield-Kuhn functor.

We can use (3.3), to compute the map ℓ_1 on homotopy groups, and we thus recover the well-known equivalences of connected covers $\mathrm{gl}_1(KU_p)_{\geq 3} \xrightarrow{\sim} (KU_p)_{\geq 3}$ and $\mathrm{gl}_1(KO_p)_{\geq 2} \xrightarrow{\sim} (KO_p)_{\geq 2}$.

To understand (3.1) in the general case, we can formally pull the operations ψ_A (which are ring homomorphisms) out of the logarithmic series, obtaining

$$\ell_n(x) = \sum_{k=0}^{n} (-1)^k p^{\binom{k}{2}} T_j(\log x) \qquad \text{where} \qquad T_j := p^{-k} \sum_{A \subseteq \Lambda^*,\ |A| = p^k} \psi_A. \tag{3.4}$$

For $x = 1 + y \in E^0(X)^\times$ such that $y^2 = 0$, this becomes

$$\ell_n(x) = \sum_{k=0}^{n} (-1)^k p^{\binom{k}{2}} T_j(y). \tag{3.5}$$

In particular, taking X to be a sphere, we obtain a formula which computes the effect of $\ell_n \colon \mathrm{gl}_1(E) \to E$ on homotopy groups (up to torsion).

To understand how we can compute these operations, we must discuss "power operations" for $K(n)$-local commutative rings (such as Morava E-theory). The short answer is that such operations are controlled by certain isogenies of the formal group associated to the theory, and in particular the operators T_j are "Hecke operators" for the theory. We will come back to this in §4.

3.3. Application of the logarithm to orientation problems.

Example 3.3. Orientations of K-theory. Consider the composite

$$spin \to o \xrightarrow{j} \mathrm{gl}_1(S) \to \mathrm{gl}_1(KO_p) \xrightarrow{\ell_1} KO_p. \tag{3.6}$$

It is a standard calculation that all maps $spin = \Sigma^{-1}(KO_{\geq 4}) \to KO_p$ are null homotopic. As ℓ_1 is here an equivalence on 2-connected covers (3.2), we immediately see that the composite of (3.6) is null-homotopic, and thus there must exist a map $MSpin \to KO_p$ of commutative S-algebras. It can be shown that the Atiyah-Bott-Shapiro orientation can be realized by one such map; see [Hop02, §6.1] for a sketch.

3.4. Application to the string orientation of tmf. With Matt Ando and Mike Hopkins, we have shown that tmf, the spectrum of **topological modular forms**, admits a commutative S-algebra map $MString \to \mathrm{tmf}$ which realizes the Witten genus. Our proof only exists in preprint form, though the result was announced in [Hop02, §6], to which the reader is referred for background. Here we will only note the way in which the logarithmic operation enters into the proof.

The key point is to understand the homotopy type of $\mathrm{gl}_1(\mathrm{tmf}_p)$, where tmf_p is the completion of tmf at a prime p. General results localizations show that there is a commutative square of spectra

$$\begin{tikzcd} \mathrm{gl}_1(\mathrm{tmf}_p) \arrow[r, "\ell_2"] \arrow[d, "\ell_1"'] & \mathrm{tmf}_{K(2)} \arrow[d, "\iota_{K(2)}"] \\ \mathrm{tmf}_{K(1)} \arrow[r, "\gamma"] & (\mathrm{tmf}_{K(2)})_{K(1)} \end{tikzcd}$$

which, after taking 3-connected covers, becomes a homotopy pullback. Both $\mathrm{tmf}_{K(2)}$ and $\mathrm{tmf}_{K(1)}$ are relatively well-understood objects: $\mathrm{tmf}_{K(2)}$ is closely related to Morava E-theory spectra at height 2, while $\mathrm{tmf}_{K(1)}$ is related to the theory of p-adic modular forms. To understand the homotopy type of $\mathrm{gl}_1(\mathrm{tmf}_p)$, we must get our hands on the map γ. It can be shown that maps $\mathrm{tmf}_{K(1)} \to (\mathrm{tmf}_{K(2)})_{K(1)}$ are characterized (up to homotopy) by their effect on homotopy groups. Thus, the key is to compute the effect of γ on homotopy groups.

Recall that there is a map $\pi_*\mathrm{tmf} \to MF_*$ to the ring of modular forms (with integer coefficients), which is an isomorphism up to torsion. Given an element in $\pi_{2k}\mathrm{tmf}$ corresponding to a modular form f of weight k, we use (3.2) to obtain

$$\ell_1(f) = f^*(q) := f(q) - p^{k-1}f(q^p),$$

where the result is stated in terms of the q-expansion of f. The series $f^*(q)$ is the q-expansion of a p-adic modular form, and thus corresponds to an element of $\pi_{2k}\mathrm{tmf}_{K(1)}$.

When we evaluate ℓ_2 at an element of $\pi_{2k}\mathrm{tmf}$ associated to a modular form f, the result turns out to be again a modular form; i.e., the image of $\ell_2 \colon \pi_*\mathrm{tmf}_p \to \pi_*\mathrm{tmf}_{K(2)}$ is contained in the image of $\iota_{K(2)} \colon \pi_*\mathrm{tmf}_p \to \pi_*\mathrm{tmf}_{K(2)}$. In fact, (3.5) implies

$$\ell_2(f) = (1 - T_1 + p^{k-1})f,$$

where T_1 is a classical Hecke operator on modular forms (usually written $T(p)$ in this context).

Using these calculations, one can deduce that $\gamma = (\iota_{K(2)})_{K(1)} \circ (\mathrm{id} - U)$, where $U \colon \mathrm{tmf}_{K(1)} \to \mathrm{tmf}_{K(1)}$ is topological lift of the **Atkin operator** on p-adic modular forms; see [Bak89] for a construction of this topological lift. This calculation provides enough control on the homotopy type of $\mathrm{gl}_1(\mathrm{tmf}_p)$ to study the set of string-orientations of tmf.

Remark 3.2. These ideas actually allow one to construct ("by hand") a spectrum map $\ell_{\mathrm{tmf}_p} \colon \mathrm{gl}_1(\mathrm{tmf}_p) \to \mathrm{tmf}_p$, so that $\iota_{K(2)} \circ \ell_{\mathrm{tmf}_p} = \ell_2$ and $\iota_{K(1)} \circ \ell_{\mathrm{tmf}_p} = (\mathrm{id} - U) \circ \ell_1$. This fact is in need of a more natural explanation.

Remark 3.3. If $f = \sum a_n q^n$ is an *eigenform* of weight k, then $(1 - T_1 + p^{k-1})f = (1 - a_p + p^{k-1})f$. In particular, if f is an *Eisenstein series*, then $\ell_2(f) = 0$, an observation which is is key to realizing the Witten genus as a string-orientation.

We note in passing that for an eigenform which is a cusp form and normalized so that $a_1 = 1$, the expression $L(f, s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$ is precisely the L-series associated to the form. The significance of this to homotopy theory remains unclear.

4. Power operations

The notion of a power operation originated in Steenrod's construction of the eponymously named operations in ordinary cohomology with coefficients in $\mathbb{F}_p$. A convenient modern formulation is in terms of structured commutative ring spectra. There are various equivalent models of such; I will not distinguish among them here, and I will call them **commutative S-algebras**; see [EKMM97] and [MMSS01] for introductions to some of these models.

A (generalized) cohomology theory $X \mapsto E^*(X)$ is represented by a spectrum E. If E is equipped with the structure of a commutative monoid in the homotopy category of spectra, then $X \mapsto E^*(X)$ takes values in graded commutative rings. For any $m \geq 0$, there is a resulting cohomology operation $x \mapsto x^m \colon E^0(-) \to E^0(-)$ defined by taking mth powers with respect to the product.

If E is equipped with the structure of a commutative S-algebra, then the mth power map admits a refinement to a "total mth power operation" of the form

$$P_m \colon E^0(X) \to E^0(X \times B\Sigma_m),$$

where $B\Sigma_m$ is the classifying space of the symmetric group on m letters. The function P_m is a multiplicative (but non-additive) natural transformation $E^0(-) \to E^0(- \times B\Sigma_m)$ of cohomology groups.

It is convenient to regard power operations as operations on the homotopy groups of commutative S-algebras. For a commutative S-algebra R, the power construction determines gives a function

$$P_m \colon \pi_0 R \to \pi_0 R^{B\Sigma_m^+}, \tag{4.1}$$

where $R^X := \underline{\mathrm{Hom}}(\Sigma^\infty X, R)$, the spectrum of maps from the suspension spectrum of a pointed space X to R. (Operations in dimensions other 0 can also be obtained, by replacing $B\Sigma_m^+$ with a suitable Thom spectrum. In the discussion below, I will concentrate on operations in dimension 0 for simplicity.) Cohomology operations for $E^*(X)$ (such as Steenrod's for ordinary cohomology) can be obtained by setting $R = E^{X_+}$, to be thought of as the ring of E-valued cochains on X.

Let us fix a commutative S-algebra E. To calculate power operations for commutative E-algebras R, one must understand the functor $R \mapsto \pi_0(R^{B\Sigma_m^+}) = R^0(B\Sigma_m)$, which in practice can be non-trivial. The best case scenario is to have a natural "Künneth isomorphism"

$$\pi_* R^{B\Sigma_m^+} \approx \pi_* R \otimes_{\pi_* E} \pi_* E^{B\Sigma_m^+}, \tag{4.2}$$

together with calculational control of the rings $\pi_* E^{B\Sigma_m^+} \approx E^{-*} B\Sigma_m$, the E-cohomology rings of symmetric groups.

This best case scenario is in fact relatively rare. It does hold for HF-algebras, where F is a field [BMMS86]. It holds also for $K(n)$-local commutative E-algebras, where E is a Morava E-theory spectrum.

4.1. Power operations in the $K(n)$-local setting.

In 1993, Hopkins and Miller perceived that a Morava E-theory spectrum must admit an essentially unique commutative S-algebra structure; the proof of this result is in [GH04]. Therefore Morava E-theories admit a theory of power operations; such operations were first considered by Ando [And95][1].

Example 4.1. The operations ψ_A appearing in (3.1) are obtained as power operations for Morava E-theory, namely as the composite

$$E^0(X) \xrightarrow{P_{p^k}} E^0(X \times B\Sigma_{p^k}) \xrightarrow{(\mathrm{id} \times Bi)^*} E^0(X \times BA^*) \xrightarrow{\chi(-, g_A)} D \otimes_{E^0} E^0(X),$$

where P_{p^k} is the total power operation for the Morava E-theory, $i \colon A^* \to \Sigma_{p^k}$ is the inclusion defined by the left-action of A^* on its underlying set, where $p^k = |A^*|$. These are examples of the operations constructed in [And95].

[1] In fact, Ando did not make use of the commutative S-algebra structure of E, which was unavailable at the time, though he does show that the operations he constructs are the same as those obtained from any commutative S-algebra structure which might exist on E.

The theory of power operations for commutative algebras over Morava E-theories is now very largely understood, based mainly on work by Ando, Hopkins, and Strickland, who were motivated by the problem of rigidifying the Witten genus to a map of spectra [AHS04], along with some contributions by the author [Rez09].

Our goal in this section is two-fold: to show (i) the homotopy groups π_*R of a $K(n)$-local commutative E-algebra R take values in a category QCoh(Def) of sheaves on a moduli problem of "formal groups and isogenies", and (ii) the category QCoh(Def) can in practice be described using a small amount of data, and in fact at small heights can be described completely explicitly. In addition to the references given below, the material in this section is developed in detail in the preprint [Rezb].

4.2. Deformations and Morava E-theory. We fix a height n formal group (commutative, one-dimensional) G_0 over a perfect field k.

Given a formal group G over a complete local ring B, a **(G_0-)deformation structure** on G/B is a pair (i, α) consisting of an inclusion $i\colon k \to B/\mathfrak{m}$ of fields and an isomorphism $\alpha\colon i^*G_0 \xrightarrow{\sim} G_{B/\mathfrak{m}}$ of formal groups over $B/\mathfrak{m}$. We write $\mathcal{D}(G/A) = \mathcal{D}_{G_0}(G/A)$ for the set of deformation structures on G/A. Note that if $g\colon A \to A'$ is a local homomorphism, there is map $g^*\colon \mathcal{D}(G/A) \to \mathcal{D}(g^*G/A')$ induced by base change.

An **isogeny** of formal groups over A is a homomorphism $f\colon G \to G'$ such that the induced map $\mathcal{O}_{G'} \to \mathcal{O}_G$ on function rings is finite and locally free; we write $\deg(f)$ for the rank of $\mathcal{O}_G$ as an $\mathcal{O}_{G'}$-module. Given such an isogeny, there is an induced pushforward map $f_!\colon \mathcal{D}(G/A) \to \mathcal{D}(G'/A)$ on sets of deformation structures, so that

$$f_*(i, \alpha) := (i \circ \phi^r, \alpha'),$$

where $\phi(a) = a^p$ is the absolute Frobenius on rings, $p^r = \deg f$, and α' is the unique isomorphism such that $\alpha' \circ F^r = f_{B/\mathfrak{m}} \circ \alpha$, where

$$F^r\colon G \to (\phi^r)^*G$$

denotes the **Frobenius isogeny**, i.e., the relative p^rth power Frobenius defined for any G over an $\mathbb{F}_p$-algebra.

Remark 4.1. An easy exercise shows that, when $\mathbb{F}_p \subseteq A$, there is an identity $F_* = \phi^*$ of maps $\mathcal{D}(G/A) \to \mathcal{D}(\phi^*G/A)$.

Given a complete local ring B, let $\mathrm{Def}^0(B) = \mathrm{Def}^0_{G_0}(B)$ denote the groupoid so that

- *objects* are pairs $(G, (i, \alpha) \in \mathcal{D}(G/B))$,
- *morphisms* are isomorphims $f\colon G \to G'$ such that $f_*(i, \alpha) = (i', \alpha')$.

Proposition 4.2 (Lubin-Tate [LT66]). *All automorphisms in* $\mathrm{Def}^0(B)$ *are identity maps (i.e.,* $\mathrm{Def}(B)$ *is equivalent to a discrete groupoid). There exists a ring* A_0 *and a natural bijection*

$$\{\textit{local homomorphisms } A_0 \to B\} \longleftrightarrow \{\textit{iso. classes of objects in } \mathrm{Def}^0(B)\}.$$

There is a (non-canonical) isomorphism $A_0 \approx \mathbb{W}_p k[\![a_1, \dots, a_{n-1}]\!]$.

The tautological object of $\mathrm{Def}^0(A_0)$ is the **universal deformation** of G_0. It is the formal group of Morava E-theory, whose existence follows from the following.

Theorem 4.3 (Morava [Mor78], Goerss-Hopkins-Miller [GH04]). *Given a formal group G_0 of height n over a perfect field, there exists an essentially unique commutative S-algebra E, which is a complex oriented cohomology theory with $\pi_* E \approx A_0[u, u^{-1}]$ with $|u| = 2$, whose formal group is the universal deformation of G_0.*

4.3. The "pile" of deformation structures.

We enlarge the groupoid $\mathrm{Def}^0(B)$ to a category $\mathrm{Def}(B)$, with the same objects, but so that

- *morphisms* are isogenies $f\colon G \to G'$ such that $f_*(i, \alpha) = (i', \alpha')$.

To each continuous homomorphism $g\colon B \to B'$ there is an associated pullback functor $g^*\colon \mathrm{Def}(B') \to \mathrm{Def}(B)$. Thus, Def defines a (pseudo)functor

$$\{\text{complete local rings}\}^{\mathrm{op}} \to \{\text{categories}\}.$$

A *stack* is a (kind of) presheaf of groupoids. The functor Def_{G_0} gives rise to a more general kind of object, namely a presheaf of categories on the opposite category of complete local rings. This more general concept demands a new name; thus, we will speak of Def_{G_0} as the **pile**[2] of deformations of G_0 and its Frobenius isogenies.

There is a category QCoh(Def) of **quasicoherent (pre)sheaves** of modules over the structure sheaf $\mathcal{O}_{\mathrm{Def}}$ of the pile Def. An object of QCoh(Def) amounts to a choice of data $\{M_B, M_g\}$ consisting of

- for each complete local ring B, a functor $M_B\colon \mathrm{Def}(B)^{\mathrm{op}} \to \mathrm{Mod}_B$, and
- for each for each local homomorphism $g\colon B \to B'$ a natural isomorphism

  $$M_g\colon B' \otimes_B M_B \Longrightarrow M_{B'} \circ g^*\colon \mathrm{Def}(B)^{\mathrm{op}} \to \mathrm{Mod}_{B'},$$

  where $g^*\colon \mathrm{Def}(B) \to \mathrm{Def}(B')$ is the functor induced by base change along g,

together with coherence data equating $M_{g'g}$ with a composition of $M_{g'}$ and M_g.

Example 4.2. Given G/B, let $\omega_B(G/B)$ denote the B-module of invariant 1-forms on G. Because 1-forms can be pulled back along isogenies, this defines an object $\omega \in \mathrm{QCoh}(\mathrm{Def})$.

Example 4.3. Given G/B, let $\deg_B(G/B) := B$. To an isogeny $f\colon G \to G'$, we associate the map $\deg_B(f)\colon \deg_B(G'/B) \to \deg_B(G/B)$ induced by multiplication by the integer $\deg(f)$. This defines an object $\deg \in \mathrm{QCoh}(\mathrm{Def})$, the **degree sheaf**.

[2] This term was suggested by Matt Ando.

4.4. Power operations and QCoh(Def). Let E denote the Morava E-theory associated to our fixed formal group G_0/k.

Recall (4.1) the total power operation $P_m\colon \pi_0 R \to \pi_0 R \otimes_{\pi_0 E} E^0 B\Sigma_m$ defined for $K(n)$-local commutative E-algebras R. The function P_m is multiplicative (i.e., $P_m(ab) = P_m(a)P_m(b)$), but not additive. We may obtain a ring homomorphism by passing to the quotient $A_m = E^0 B\Sigma_m / I_{\mathrm{tr}}$ of $E^0 B\Sigma_m$ by the ideal $I_{\mathrm{tr}} \subseteq E^0 B\Sigma_m$ generated by the image of all transfers maps from inclusions of the form $\Sigma_i \times \Sigma_{m-i} \subset \Sigma_m$ with $0 < i < m$. The composite map

$$\overline{P}_m\colon \pi_0 R \xrightarrow{P_m} \pi_0 R^{B\Sigma_m^+} \approx \pi_0 R \otimes_{\pi_0 E} E^0 B\Sigma_m \to \pi_0 R \otimes_{\pi_0 E} E^0 B\Sigma_m / I_{\mathrm{tr}} \tag{4.3}$$

is a ring homomorphism.

The key to understanding power operations are the following result due to Strickland. To state it, it is useful to note that we can form a quotient category $\mathrm{Def}(B)/{\sim}$ of $\mathrm{Def}(B)$ by formally identifying isomorphic objects (possible exactly because there are no non-trivial automorphisms in this category), and the projection functor is an equivalence of categories.

We write $\mathrm{Def}^r(B)/{\sim}$ for the set of morphisms which correspond to isogenies of degree p^r. It is straightforward to show that elements of $\mathrm{Def}^r(B)/{\sim}$ are in bijective correspondence with pairs (G, H), where G is an object of $\mathrm{Def}^0(B)/{\sim}$ and $H \le G$ is a finite subgroup scheme of rank p^k; the correspondence sends an isogeny to its kernel.

Theorem 4.4 (Strickland [Str97], [Str98]). *There exist complete local rings A_r (for $r \ge 0$), finite and free as A_0-modules, so that*

$$\{\textit{local homomorphisms } A_r \to B\} \longleftrightarrow \mathrm{Def}^r(B)/{\sim}\,.$$

Furthermore, there is a natural identification of rings

$$E^0 B\Sigma_{p^r}/I_{\mathrm{tr}} \approx A_r.$$

As a consequence, the functor $B \mapsto \mathrm{Def}(B)$ from complete local rings to (graded) categories is represented by a graded affine category object $\{A_r\}$ in (complete local rings)$^{\mathrm{op}}$. Thus, QCoh(Def) is equivalent to a category of *comodules*, whose objects are A_0-modules M equipped with module maps

$$\psi_r\colon M \to {}^t A_r{}^s \otimes_{A_0} M,$$

which satisfy an evident coassociativity property. (There are ring maps $s, t\colon A_0 \to A_r$ corresponding to "source" and "target" in the graded category; we use superscripts to indicate the corresponding A_0-module structures on A_r.) Furthermore, the power operation maps $\overline{P}_{p^r}$ of (4.3) make $\pi_0 R$ into a comodule; i.e., (4.4) refines π_0 to a functor

$$\underline{\pi}_0\colon h\mathrm{Com}(E)_{K(n)} \to \mathrm{QCoh}(\mathrm{Def})$$

from the homotopy category of $K(n)$-local commutative E-algebra spectra, to the category of quasi-coherent sheaves of modules on Def, so that the value of $\underline{\pi}_0(R)$ at the universal deformation in $\mathrm{Def}(A_0)$ precisely the ring $\pi_0 R$.

Remark 4.5. The existence of the functor $\underline{\pi}_0$ is essentially an observation of Ando, Hopkins, and Strickland (see [AHS04]). A construction is given in [Rez09].

4.5. Additional structure. The functor $\underline{\pi}_0$ to sheaves on Def admits several additional refinements, which we pass over quickly. (Most are discussed in [Rez09]; see [BF] for a treatment of completion.)

- QCoh(Def) is a symmetric monoidal category, and $\underline{\pi}_0$ naturally takes values in QCoh(Def, Com), the category of sheaves of commutative rings in QCoh(Def).
- There is an extension to a functor $\underline{\pi}_*\colon h\mathrm{Com}(E)_{K(n)} \to \mathrm{QCoh}^*(\mathrm{Def}, \mathrm{Com})$, where the target is a category of *graded* ring objects in QCoh(Def).
- The output of $\underline{\pi}_*$ is (in a suitable sense) complete with respect to the maximal ideal of A_0.
- The rings $\underline{\pi}_0 R$ satisfy a **Frobenius congruence**. We say that an object $M \in \mathrm{QCoh}(\mathrm{Def}, \mathrm{Com})$ satisfies this condition if, for all formal groups G/B with $\mathbb{F}_p \subseteq B$, the map

  $$B^{\phi} \otimes_B M_B(G, d) \approx M_B(\phi^* G, \phi^*(d)) = M_B(\phi^* G, M_*(d)) \xrightarrow{F^*} M_B(G, d)$$

  coincides with the relative pth power map on the ring $M_B(G, d)$, for any $d \in \mathcal{D}(G/B)$.

  In terms of the comodule formulation of QCoh(Def, Com), this amounts to saying that the composite

  $$M \xrightarrow{\psi_1} {A_1}^s \otimes_{A_0} M \xrightarrow{\gamma\otimes\mathrm{id}} (A_0/p) \otimes_{A_0} M = M/p$$

  is the pth power map on M, where $\gamma\colon A_1 \to A_0/p$ is the map representing $\mathrm{Def}^0 \to \mathrm{Def}^1$ sending $G \mapsto (F\colon G \to \phi^* G)$.
- The Frobenius congruence for $\underline{\pi}_0 R$ is *witnessed* by a non-additive operation on π_0. To state this, note that there is A_0-module homomorphism $\epsilon\colon A_1 \to A_0$ lifting $\gamma\colon A_1 \to A_0/p$. Using this, we define a homomorphism of abelian groups

  $$Q\colon \pi_0 R \xrightarrow{\psi_1} A_1 \otimes_{A_0} \pi_0 R \xrightarrow{\epsilon\otimes\mathrm{id}} A_0 \otimes_{A_0} \pi_0 R = \pi_0 R,$$

  which satisfies $Q(x) \equiv x^p \mod p$. The *witness* is a (non-additive) natural operation $\theta\colon \pi_0 R \to \pi_0 R$ satisfying $Q(x) = x^p + p\theta(x)$.

Remark 4.6. Mathew, Naumann, and Noel have observed [MNN] that the mere existence of a witness for the Frobenius congruence allows one to show that any p^r-torsion element in the homotopy of a $K(n)$-local commutative E-algebra is nilpotent. Using this together with the nilpotence theorem of Devinatz-Hopkins-Smith, they prove a conjecture of May: for any commutative S-algebra R the kernel of the Hurewicz map $\pi_* R \to H_*(R, \mathbb{Z})$ consists of nilpotent elements.

The outcome of the additional structure outlined above is that there exists a refinement of the homotopy functor $\pi_*\colon h\mathrm{Com}(E)_{K(n)} \to \mathrm{Mod}(E_*)$ to a functor

$$\underline{\pi}_*\colon h\mathrm{Com}(E_{G_0})_{K(n)} \to \mathcal{T}_{G_0}$$

to a certain algebraically defined category $\mathcal{T}_{G_0}$, whose construction depends only on the formal group G_0/k.

Example 4.4. If $G_0/k = G_m/\mathbb{F}_p$, the formal multiplicative group, then $E_{G_0} = KU_p$ is p-complete complex K-theory. The category $\mathcal{T}_{G_0}$ is the category of p-complete $\mathbb{Z}/2$-graded θ^p-rings described by Bousfield [Bou96].

4.6. The quadratic nature of QCoh(Def).

Remarkably, making calculations about objects in QCoh(Def) is far more tractable than the above suggests. This is because the representing coalgebra $\{A_r\}$ is "quadratic". This means the following: an object of QCoh(Def) is determined, up to canonical isomorphism, by its underlying module M and the structure map $\psi_1\colon M \to A_1 \otimes_{A_0} M$, which is subject to a single relation, namely that there *exists* a dotted arrow in the following diagram of $\pi_0 E$-modules:

$$\begin{tikzcd} & M \arrow[rr, "\psi_1"] \arrow[d, dotted] & & {}^tA_1{}^s \otimes_{A_0} M \arrow[d, "\mathrm{id}\otimes\psi_1"] \\ & {}^tA_2{}^s \otimes_{A_0} M \arrow[rr, "\nabla\otimes\mathrm{id}"] & & {}^tA_1{}^s \otimes_{A_0} {}^tA_1{}^s \otimes_{A_0} M \end{tikzcd} \tag{4.4}$$

where ∇ encodes composition of two morphisms of degree p in Def. In particular, the category QCoh(Def) can be reconstructed using only knowledge of the rings $\pi_0 E$, A_1, and A_2, and the ring homomorphisms s, t, and ∇.

Example 4.5. *Multiplicative group.* For $G_0/k = G_m/\mathbb{F}_p$, the rings $A_r \approx \mathbb{Z}_p$ for all r. An object in QCoh(Def) amounts to a $\mathbb{Z}_p$-module M equipped with an endomorphism $\psi\colon M \to M$.

Example 4.6. *Height 2.* When E is associated to a formal group of height 2, it is possible to give a completely explicit description of QCoh(Def), by using explicit formulas obtained from the theory of elliptic curves. For instance, let G_0 be the formal completion of the supersingular elliptic curve over $\mathbb{F}_2$. In this case, $A_0 = \mathbb{Z}_2[\![a]\!]$, $A_1 = \pi_0 E[d]/(d^3 - ad - 2)$, and the ring homomorphisms $s, t\colon A_0 \to A_1$ are given by $s(a) = a$ and $t(a) = a^2 + 3d - ad^2$. The ring A_2 is the pullback of

$$A_1{}^t \otimes_{A_0} {}^sA_1 \xrightarrow{w\otimes\mathrm{id}} A_1 \xleftarrow{s} A_0$$

where $w\colon A_1 \to A_1$ sends $w(a) = t(a)$ and $w(d) = a - d^2$. (The map w classifies the operation of sending a p-isogeny of elliptic curves to its dual isogeny.) The map Δ is the evident inclusion map. The above description is outlined (admittedly in very rough form) in [Reza]. Zhu [Zhu14] has calculated a similar example at the prime 3.

Example 4.7. *Height 2, modulo p.* It is possible to give a uniform description of this structure at height 2, if we work modulo the prime. Fix a supersingular elliptic curve C_0 over $\mathbb{F}_p$ whose Frobenius isogeny satisfies $F^2 = -p$ (such always exist), and let G_0 be its formal completion. Then, following an observation of [KM85, 13.4.6], we see that $A_1/p \approx \mathbb{F}_p[\![a_1, a_2]\!]/(\,(a_1^p - a_2)(a_1 - a_2^p)\,)$, so that $s, t\colon A_0/p = \mathbb{F}_p[\![a]\!] \to A_1/p$ send $s(a) = a_1$ and $t(a) = a_2$; the rings A_r/p can be described similarly. See [Rez12], especially §3.

In the general case, the quadraticity of QCoh(Def) is a consequence of a stronger theorem: that the algebra of power operations for Morava E-theory is Koszul.

4.7. The ring of power operations is Koszul.

We observe that QCoh(Def) is equivalent to a category of modules over an associative ring $\Gamma := \bigoplus \mathrm{Hom}_{A_0}(A_r, A_0)$. In particular, it is an abelian category with enough projectives and injectives. In their work, Ando, Hopkins, and Strickland perceived that the ring Γ should have finite homological dimension (see discussion at the end of §14 in [Str97]). That this is so is a consequence of the following theorem.

Theorem 4.7 ([Rezc]). *The ring Γ is Koszul, and thus objects of* $\mathrm{QCoh}(\mathrm{Def}_{G_0})$ *admit a functorial resolution by a "Koszul complex". More precisely, there is a functor $\mathcal{C}\colon$ QCoh(Def) $\to$ Ch(QCoh(Def)) together with a natural augmentation $\epsilon\colon \mathcal{C}(M) \to M$ which is a quasi-isomorphism if M is projective as a $\pi_0 E$-module. Furthermore,*

$$\mathcal{C}_k(M) = \Gamma \otimes_{\pi_0 E} C_k \otimes_{\pi_0 E} M,$$

where C_k is a $\pi_0 E$-bimodule which is (i) free and finitely generated as a right $\pi_0 E$-module, and (ii) $C_k = 0$ if $k > n$, where n is the height of the formal group G_0.

As a consequence, Γ has global dimension $2n$, where n is the height of the formal group.

Remark 4.8. The proof of (4.7) given in [Rezc] is purely topological, making no reference to the interpretation of QCoh(Def) in terms of isogenies of formal groups. The proof is inspired by the theory of the Goodwillie calculus of the identity functor on spaces, and in particular by the work of Arone-Mahowald [AM99] on the $K(n)$-local homotopy type of the layers of the Goodwillie tower of the identity functor evaluated at odd spheres. They show that the $K(n)$-local homotopy type of an odd sphere is concentrated purely at layers p^k for $0 \le k \le n$.

Remark 4.9. The statement of (4.7) is purely a statement about deformations about formal groups and their isogenies, and thus should in particular admit a proof which does not use topology. I do not know such a proof in general, but such a purely algebro-geometric proof exists in the cases $n = 1$ and $n = 2$; see [Rez12] for the height 2 case.

Remark 4.10. Ando, Hopkins, and Strickland originally conjectured a particular form for a finite complex such as that in terms of (4.7), in terms of a "Tits building" associated to subgroups of $\mathbb{G}_E[p]$, the p-torsion subgroup of the formal group $\mathbb{G}_E$. Their original complex in the height 2 case can be constructed using the arguments of [Rez12]. Recently, Jacob Lurie has shown how (4.7) can be used to recover the original proposal of Ando, Hopkins, and Strickland.

4.8. Computing maps of $K(n)$-local E-algebras. We can use the theory described above to describe the E_2-term of a spectral sequence computing the space of maps between commutative E-algebras. We describe how this works in a special case.

Let R, F be two $K(n)$-local commutative E-algebras equipped with an augmentation to E. There is a spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\mathrm{Com}(E)^{\mathrm{aug}}_{K(n)}(R,F) \tag{4.5}$$

computing homotopy groups of the derived space of maps in the category of augmented $K(n)$-local commutative E-algebras. When π_*R is *smooth* over π_*E, the E_2-term takes the form

$$E_2^{s,t} \approx \begin{cases} \mathcal{T}_{/E_*}(\pi_*R, \pi_*F) & \text{if } (s,t)=(0,0), \\ \mathrm{Ext}^s_{\mathrm{QCoh(Def)}}(\omega^{-1/2}\otimes \widehat{Q}(\pi_*R), \omega^{(t-1)/2}\otimes \pi_*\overline{F}) & \text{otherwise.} \end{cases} \tag{4.6}$$

Here $\widehat{Q}(\pi_*R)$ is the module of indecomposables of the augmented ring π_*R (completed with respect to the maximal ideal of π_*E), $\pi_*\overline{F}$ is the augmentation ideal of π_*F, and ω is the module of invariant 1-forms (4.2). In this situation, the spectral sequence is strongly convergent, and is non-zero for only finitely many values of s.

5. Tangent spaces to cochains and $\Phi_n(S^{2d-1})$

5.1. Derived indecomposables of commutative ring spectra. Fix a commutative ring A, and consider an augmented commutative A-algebra R; i.e., an A-algebra equipped with an A-algebra map $\pi\colon R \to A$. We can consider the A-module $T^*_{A,\pi}(R) := I/I^2$ of **indecomposables** with respect to the augmentation, where $I = \mathrm{Ker}(\pi)$. In geometrical terms, $T^*_{A,\pi}(R)$ is the cotangent space to $\mathrm{Spec}(R)$ at the point corresponding to π. The dual module $T_{A,\pi}(R) := \underline{\mathrm{Hom}}_A(T^*_{A,\pi}(R), A)$ can be viewed as a tangent space at π.

This cotangent space construction admits a derived generalization to commutative ring spectra. Given a commutative S-algebra A, and an augmented commutative A-algebra R, there is an A-module of **derived indecomposables** constructed by Basterra [Bas99], and which we will also denote by $T^*_A(R)$ (taking the map π to be understood).[3] We write $T_A(R) := \underline{\mathrm{Hom}}_A(T^*_A(R), A)$ for the corresponding "tangent space".

We note two alternate descriptions of these constructions, which follow from work of Basterra and Mandell ([BM05], especially §2).

- The cotangent functor $T^*_A\colon \mathrm{Com}^{\mathrm{aug}}_A \to \mathrm{Mod}_A$ is a kind of stabilization functor. It is most conveniently expressed in terms of an equivalence $\mathrm{Com}^{\mathrm{aug}}_A \approx \mathrm{Com}^{\mathrm{nu}}_A$ between augmented and non-unital algebras, so that

$$T^*_A(R) \approx \mathrm{hocolim}_n\, \Omega^n_{\mathrm{nu}}\Sigma^n_{\mathrm{nu}}(I),$$

[3] This functor is also called "reduced topological Andre-Quillen cohomology".

where I is the homotopy fiber of the augmentation $R \to A$ viewed as a non-unital algebra, and Σ_{nu} and Ω_{nu} are loop and suspension functors in the homotopy theory of $\mathrm{Com}_A^{\mathrm{nu}}$.

- The tangent space module can also be described as "functions to the dual numbers". That is, the underlying spaces of the spectrum $T_A(R)$ can be identified as
$$\Omega^{\infty+t}T_A(R) \approx \mathrm{Com}_A^{\mathrm{aug}}(R, A \ltimes \Omega^t A),$$
where $A \ltimes \Omega^t A$ is a split square-zero extension of A by a shift of A.

Given a based space X and a commutative S-algebra A, we can apply these constructions to the cochain algebra $A^{X_+} := \underline{\mathrm{Hom}}(\Sigma^\infty(X_+), A)$, which is a commutative A-algebra equipped with an augmentation corresponding to the basepoint of X.

Example 5.1. Take $A = H\mathbb{Q}$, the rational Eilenberg-Mac Lane spectrum. For spaces X which are simply connected and of finite-type, we have a natural isomorphism
$$\pi_* T_{H\mathbb{Q}}(H\mathbb{Q}^{X_+}) \approx \pi_*(X) \otimes \mathbb{Q}$$
from the homotopy of the tangent space spectrum to the rational homotopy groups of X. This is a modern restatement of a well-known fact of rational homotopy theory (e.g., [Sul77, Thm. 10.1]).

Example 5.2. Take $A = H\overline{\mathbb{F}}_p$, the Eilenberg-Mac Lane spectrum of the algebraic closure of $\mathbb{F}_p$. Then $T_{H\overline{\mathbb{F}}_p}(H\overline{\mathbb{F}}_p^{X_+}) \approx 0$ by [Man06, Prop. 3.4]. Mandell's work shows that the cochain spectrum $H\overline{\mathbb{F}}_p^{X_+}$ contains complete information about the mod p homotopy type of simply connected finite-type X, but this information cannot be extracted from the tangent space.

It turns out that for $K(n)$-local rings, the (co)tangent space behaves more like rational homology than mod p-homology, where the role of rational homotopy groups is replaced with the Bousfield-Kuhn functor.

5.2. The tangent space to cochains for $K(n)$-local rings.

Let A be a $K(n)$-local commutative S-algebra. For a based space X, we can construct **comparison maps** which relate the A cohomology/homology of the spectrum $\Phi_n X$ with the tangent/cotangent space of the cochain ring A^{X_+}. These take the form
$$c_X^*\colon T_A^*(A^{X_+}) \to \underline{\mathrm{Hom}}_A(\Phi_n X, A) \quad \text{and} \quad c_X\colon A \wedge \Phi_n X \to T_A(A^{X_+}).$$

Remark 5.1. Here is an idea of how to build c_X^* (the map c_X is obtained by taking A-linear duals). Given a space X, apply Φ_n to the tautological map $u\colon X \to \Omega^\infty\Sigma^\infty X$, obtaining
$$\Phi_n X \xrightarrow{\Phi_n(u)} \Phi(\Omega^\infty\Sigma^\infty X) \approx (\Sigma^\infty X)_{K(n)}.$$

Taking functions into a $K(n)$-local ring A gives

$$\kappa_X\colon A^X \approx \underline{\mathrm{Hom}}((\Sigma^\infty X)_{K(n)}, A) \to \underline{\mathrm{Hom}}(\Phi_n X, A).$$

The object A^X is the augmentation ideal of A^{X_+}, and its stabilization as a non-unital A-algebra is $T_A^*(A^{X_+})$. The map c_X^* is constructed as the limit of the collection of maps

$$\Sigma^n_{\mathrm{nu}}(A^X) \to A^{\Omega^n X} \xrightarrow{\kappa_X} \underline{\mathrm{Hom}}_A(\Phi_n(\Omega^n X), A) \approx \Omega^{-n}\underline{\mathrm{Hom}}_A(\Phi_n X, A),$$

where we have used the fact that Φ_n commutes with Ω up to weak equivalence.

Mark Behrens and I have recently proved the following result, which shows the comparison map is an isomorphism for odd-dimensional spheres.

Theorem 5.2. *For any $K(n)$-local commutative ring A, and $X = S^{2d-1}$, the maps c_X and c_X^* induce isomorphisms in $K(n)$-homology.*

Remark 5.3. A consequence of the proof is a natural identification of E_*-modules

$$C_k \approx E_*^\wedge \partial_{p^k}(S^1)_{K(n)},$$

where C_k is the module in the Koszul resolution of (4.7), with the E-homology of the p^kth layer of the Goodwillie tower of the identity functor, evaluated at the circle S^1.

Remark 5.4. Combining (5.2) with remarks from §5.1, we see that we can use the spectral sequence (4.5) to compute $\pi_*(E \wedge \Phi_n(S^{2d+1})$. By (4.6), the E_2-term is

$$E_2^{s,t} \approx \mathrm{Ext}^s_{\mathrm{QCoh(Def)}}(\omega^{d-1}, \omega^{(t-1)/2} \otimes \mathrm{nul}) \Longrightarrow E^\wedge_{t-s}\Phi_n S^{2d-1},$$

where $\mathrm{nul} \in \mathrm{QCoh(Def)}$ is the object corresponding to the comodule $M = A_0$ whose coaction maps $\psi_r\colon M \to A_r \otimes_{A_0} M$ are identically 0.

Explicit calculations show that, for $n = 1, 2$, the only non-vanishing groups are when $s = n$, and thus this gives a complete calculation in that case. For $n = 1$, this recovers calculations of Bousfield [Bou99]. Details are provided in [Rezb].

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