# RINGS OF POWER OPERATIONS FOR MORAVA $E$-THEORIES ARE KOSZUL

CHARLES REZK

Abstract. We show that the ring of power operations for any Morava $E$-theory is Koszul.

## 1. Introduction

1.1. **Power operations for commutative ring spectra.** Given a structured commutative ring spectrum $E$, it is an important task to understand its theory of *power operations*. For this paper, power operations are *additive* operations on the homotopy groups of commutative $E$-algebras which arise as the residue of the (non-linear) multiplicative structure of a structured commutative ring spectrum. The closest classical analogue to power operations is the Frobenius map on commutative rings of finite characteristic; indeed there is a close connection between power operations and the Frobenius map. The most familiar examples of power operations are the Steenrod operations on the mod $p$ homology of a space, which are in fact a specialization of more general operations defined on the homotopy of a commutative $H\mathbb{F}_p$-algebra, where $H\mathbb{F}_p$ denotes the mod $p$ Eilenberg-Mac Lane spectrum.

1.2. **Power operations for Morava $E$-theory.** In this paper, we address the ring of power operations for a Morava $E$-theory spectrum. Recall that to each formal group $G$ of height $n$ defined over a perfect field $k$ of characteristic $p$, there is an associated **Morava $E$-theory spectrum**, an even periodic complex orientable theory whose associated formal group is the *universal deformation* of $G$ (as constructed by Lubin and Tate). The coefficient ring of $E$ has the form

$$\pi_* E = \mathbb{W}k[\![u_1, \dots, u_{n-1}]\!][u^{\pm}]$$

where $\mathbb{W}k$ is the unramified extension of $\mathbb{Z}_p$ with residue field $k$, $u_1, \dots, u_{n-1} \in \pi_0 E$, and $u \in \pi_2 E$. Note that $\pi_0 E$ is a complete local ring. By a theorem of Goerss, Hopkins, and Miller [GH04], every Morava $E$-theory spectrum admits the structure of a commutative $S$-algebra in an essentially unique way.

Fix $E$ to be a Morava $E$-theory spectrum. Let $\mathrm{Alg}_E$ denote the category of commutative $E$-algebras, and let $\mathrm{Alg}_{E,K(n)}$ denote the category of $K(n)$-*local* commutative $E$-algebras. The **ring of (additive) $E$-theory power operations** $\Gamma$ is a set of natural additive operations on $\pi_0$ of $K(n)$-local commutative $E$-algebras. In fact, the completion $\Gamma^\wedge_{\mathfrak{m}}$ of $\Gamma$ at the maximal ideal of $\pi_0 E$ is precisely the endomorphism ring of the homotopy functor $\pi_0 \colon h\mathrm{Alg}_{E,K(n)} \to \mathrm{Ab}$. (We give the definition of $\Gamma$ in 3.12, and describe its relation to additive power operations in (3.22).)

*Date*: July 10, 2017.

In this paper, we will need to consider a variant of $\Gamma$. Given an *augmented* $K(n)$-local commutative $E$-algebra, we may consider a set $\Delta$ of natural additive operations on the *indecomposable quotient* of $\pi_0$. The completion $\Delta^\wedge_{\mathfrak{m}}$ at the maximal ideal of $\pi_0 E$ is the endomorphism ring of the indecomposables functor $Q(\pi_0)\colon \mathrm{Alg}_{E,K(n)/E} \to \mathrm{Ab}$, i.e., it is a ring of operations acting on the "cotangent space" of the homotopy of a commutative $E$-algebra at the augmentation. There is an isomorphism of rings $\Gamma \approx \Delta$ (3.20).

1.3. *Remark.* If $X$ is any space, then the spectrum $E^{X_+}$ of functions from the suspension spectrum of $X$ to $E$ is naturally a $K(n)$-local $E_\infty$-ring spectrum. Thus, the cohomology group $E^0(X) = \pi_0(E^{X_+})$ is naturally equipped with the structure of a $\Gamma$-module, so $\Gamma$ gives rise to a family of unstable cohomology operations for $E$.

1.4. *Example.* The simplest example of a Morava $E$-theory is complex $K$-theory completed at a prime $p$, which is associated to the multiplicative group law (of height 1) over $\mathbb{F}_p$. In this case $\Gamma \approx \mathbb{Z}_p[\psi^p]$, where $\psi^p$ is a power operation which on $K$-theory cohomology rings gives the $p$th Adams operation.

There is a *non-additive* operation $\theta^p$ which acts on $\pi_0$ of a $K(1)$-local commutative $K$-algebra, which is characterized by the identity $\psi^p(x) = x^p + p\theta^p(x)$. Although $\theta^p$ is not additive, a consequence of this identity is that $\theta^p$ is additive *modulo decomposables.* It turns out that $\Delta \approx \mathbb{Z}_p[\theta^p]$.

1.5. *Remark.* Explicitly, $\Gamma = \bigoplus_{k\geq 0} \Gamma[k]$ is a graded ring with

$$\Gamma[k] \approx \mathrm{Ker}\big[E_0^\wedge B\Sigma_{p^k} \to \bigoplus_{0<j<p^k} E_0^\wedge B(\Sigma_j \times \Sigma_{p^k-j})\big],$$

the map being induced by the transfer associated to the inclusions $\Sigma_j \times \Sigma_{p^k-j} \subset \Sigma_{p^k}$ (3.12). Here $E_*^\wedge(X)$ denotes the *completed $E$-homology* of $X$ (3.4). Similarly, $\Delta = \bigoplus_{k\geq 0} \Delta[k]$ is a graded ring with

$$\Delta[k] \approx \mathrm{Cok}\big[\bigoplus_{0<j<p^k} E_0^\wedge B(\Sigma_j \times \Sigma_{p^k-j}) \to E_0^\wedge B\Sigma_{p^k}\big],$$

the map being induced by the evident inclusions of groups (3.14).

A key result, due to Strickland [Str98] is that each $\Gamma[k]$ and $\Delta[k]$ is a finitely generated free $E_0$ module (3.13).

1.6. *Remark.* For a Morava $E$-theory associated to a height $n$-formal group $G/k$, its ring $\Gamma$ of power operations is entirely determined by the formal group $G/k$. In particular, the category of modules over $\Gamma$ can be identified with a certain category of quasi-coherent sheaves on a stack of "deformations of $G/k$ and isogenies which lift powers of Frobenius". This identification is an observation of Ando, Hopkins, and Strickland (it is prefigured in [AHS04], and an explicit statement is given [Rez09, Thm. B]). We won't need this identification in this paper.

The ring $\Gamma$ of *additive* power operations for $E$ is just one piece of the story of its power operations, which are controlled by a certain monad $\mathbb{T}$ on the category of $E_*$-modules. This monad was introduced in [Rez09]; we will review some of its properties in (3.5).

1.7. **Koszul algebras and the main theorem.** The notion of Koszul algebra was introduced by Priddy [Pri70]. Roughly speaking, a Koszul algebra is a graded $k$-algebra $A$ which admits a *Koszul complex*, namely a functorial resolution $\mathcal{C}_\bullet(M) = \big(\cdots \to A \otimes_k C_n \otimes_k M \to A \otimes_k C_{n-1} \otimes_k M \to \cdots\big)$ of left $A$-modules, which is "minimal", in the sense that $k \otimes_A \mathcal{C}_\bullet(M)$ has trivial differentials, and thus $C_n = \operatorname{Tor}_n^A(k, M)$ (if $M$ is flat over $k$).

Our main result is the following.

**Main Theorem.** *The rings* $\Gamma$ *and* $\Delta$ *of power operations for any Morava* $E$*-theory, are Koszul with respect to their natural grading* (1.5).

A subtlety is that although $\Gamma$ contains the coefficient ring $E_0 = \pi_0 E$ of the theory $E$, this subring is not central in $\Gamma$. The notion of Koszul we will use (described in §4) will make sense for such rings. Furthermore, with this definition, it will be a *consequence* of the main theorem that the ring $\Gamma$ is *quadratic* (4.10), i.e., there is an isomorphism $\Gamma \approx T_{E_0}(\Gamma[1])/(R)$ with a quotient of the tensor algebra on the degree 1 part by relations $R \subseteq \Gamma[2]$ contained in the degree 2 part.

Furthermore, a straightforward calculation of ranks (8.8) shows that, for a Morava $E$-theory of height $n$, the objects $C_k$ satisfy $C_k = 0$ when $k > n$. That is, the ring of power operations for Morava $E$-theory of height $n$ has a *bounded* Koszul resolution, of length $n+1$.

The proof of the main theorem is given in (8.7).

1.8. **Some consequences of the theorem.** We briefly list some applications and consequences of this result.

*Description of* $\gamma$ *at height 2.* As noted, a consequence of our theorem is that the ring $\Gamma$ is quadratic with respect to the grading $\Gamma = \bigoplus \Gamma[k]$: i.e., it is generated by $\Gamma[1]$ with relations given by the kernel of multiplication $\Gamma[1] \otimes_{E_0} \Gamma[1] \to \Gamma[2]$. Furthermore $\Gamma[1]$ and $\Gamma[2]$, by a theorem of Strickland [Str97], are as $E_0$-modules dual to rings which classify subgroup divisors of order $p$ and $p^2$ of the formal group of $E$. Thus, an explicit calculation of the structure of $\Gamma$ can be recovered from an explicit understanding of these subgroup rings.

As noted above, the height 1 case is well known (at every prime $\Gamma$ is a polynomial ring on one generator). An explicit description of $\Gamma$ at height 2 by the author at the prime 2 [Rez08], and by Yifei Zhu at primes 3 and 5 [Zhu14]. Furthermore, Zhu has described a uniform procedure for calculating the height 2 algebra at all primes [Zhu15].

*Spaces of maps between commutative* $E$*-algebras.* Homological algebra associated to $\Gamma$ appears as input to spectral sequences and obstruction theory which compute maps between $K(n)$-local commutative ring spectra. These tools are a special case of the general machinery of Goerss and Hopkins [GH04].

Here is an example of how our theorem aids such calculations (details are in the preprint [Rez13]). Fix a height $n$ Morava $E$-theory. Suppose $R$ and $F$ are $K(n)$-local commutative $E$-algebras equipped with an augmentation to $E$. Then there is a spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s} \operatorname{Map}(R, F)$$

computing homotopy groups of the space of maps of augmented $E$-algebras. The $E_2$-term is Quillen cohomology of the $\mathbb{T}$-algebra $\pi_* R$ with coefficients in $\pi_* F$.

When $\pi_* R$ is *smooth* as a $\pi_* E$-algebra, a composite functor spectral sequence an isomorphism of the $E_2$-term with

$$E_2^{s,t} = \operatorname{Ext}^s_\Delta(\widehat{Q}(\pi_* R), \pi_* \overline{F}).$$

This is Ext in graded $\Delta$-modules, where $\widehat{Q}$ denotes the completion of the indecomposables of $\pi_*R$ with respect to the maximal ideal of $E_*$, and $\overline{F}$ is the fiber of the augmentation $F \to E$.

When $\pi_*R$ is smooth over $\pi_*E$, the indecomposables $\widehat{Q}(\pi_*R)$ are projective over $\pi_*E$. Recall that $\Delta$ (the ring of operations which acts naturally on indecomposables) is isomorphic to $\Gamma$, and hence is Koszul by (1.7). Our theorem (1.7) then tells us that $E_2^{s,*} = 0$ for $s > n$, and furthermore provides a Koszul resolution for computing the $E_2$-term. In particular, makes possible explicit calculations at height 2, some of which are described in the preprint [Rez13].

1.9. **Sketch of the proof.** We briefly indicate here the structure of the proof, which occupies the rest of the paper, and is completed in (8.7). We fix a Morava $E$-theory spectrum $E$ for some height $n \geq 1$. The argument will not apply directly to the ring $\Gamma$ of additive power operations for $E$, but rather to the ring $\Delta$ mentioned earlier, which is isomorphic to $\Gamma$ as a graded ring by (3.20).

Consider the functor $\widetilde{C}$ (2.5) which associates to a space $X$ the free non-unital $E_\infty$-algebra on $X$:

$$\widetilde{C}(X) \approx \coprod_{m\geq 1} X^m_{h\Sigma_m}.$$

There is an analogous functor $\widetilde{D}$ (2.6) on spectra, given by

$$\widetilde{D}(Y) \approx \bigvee_{m\geq 1} Y^{\wedge m}_{h\Sigma_m},$$

and we have that $\Sigma^\infty_+\widetilde{C} \approx \widetilde{D}\Sigma^\infty_+$.

In general, given a functor $F$ from an additive category to an abelian category, we can define a linearization (5.1) of $F$, by

$$\mathcal{L}_F(X) \overset{\text{def}}{=} \operatorname{Cok}\left[F(p_1+p_2) - F(p_1) - F(p_2) \colon F(X\oplus X) \to F(X)\right],$$

where $p_1, p_2 \colon X \oplus X \to X$ are the two projections.

Let $E^\wedge_* X$ denote the completed $E$-homology ((3.4) of a space $X$. Applying linearization to the composite functor $F = E^\wedge_* D$, with $X = S^0$, we are lead to consider the cokernel of a map

$$E^\wedge_* D(S^0 \vee S^0) \to E^\wedge_* D(S^0)$$

The first step (5.7) of the proof is to identify this cokernel (the linearization of $F$ at $S^0$) with the underlying $E_*$-module of an algebra $\Delta$ of power operations (see §3). There is a decomposition $\Delta \approx \bigoplus_{k\geq 0} \Delta[k]$, where $\Delta[k]$ comes from the $m = p^k$ summand in $C$. (We actually state and prove this step in terms of an algebraic approximation functor $\widetilde{\mathbb{T}}$ (§3.5), which has the property that $\widetilde{\mathbb{T}}(\bigoplus E_*) \approx E^\wedge_* \widetilde{D}(\bigvee S^0)$.)

Because $\widetilde{D}$ is a monad on the homotopy category of spectra, we may consider the two-sided bar construction $\mathcal{B}(\widetilde{D}) = \mathcal{B}(\widetilde{D}, \widetilde{D}, \widetilde{D})$. A similar argument (5.8) identifies the cokernel of the analogous map

$$E^\wedge_* \mathcal{B}(\widetilde{D})(S^0 \vee S^0) \to E^\wedge_* \mathcal{B}(\widetilde{D})(S^0)$$

(i.e., the linearization of $F = E^\wedge_* \mathcal{B}(\widetilde{D})$ at $S^0$) with the two-sided bar construction $\mathcal{B}(\Delta) = \mathcal{B}(\Delta, \Delta, \Delta)$ of the ring $\Delta$. (Again, our actual statement is given in terms of the bar construction $\mathcal{B}(\widetilde{\mathbb{T}})$ of the algebraic approximation functor $\widetilde{\mathbb{T}}$.)

What we are actually interested in is a certain quotient $\overline{\mathcal{B}}(\Delta)$ of $\mathcal{B}(\Delta)$, which is isomorphic to $\mathcal{B}(E_*, \Delta, E_*)$. This quotient admits (4.1) a decomposition $\bigoplus_{k\geq 0} \overline{\mathcal{B}}(\Delta)[k]$ associated to the decomposition of $\Delta$. We observe (4.4) that $\Delta$ is Koszul if and only if the homology of $\overline{\mathcal{B}}(\Delta)[k]$ is concentrated in degree $k$, for all $k \geq 0$. More precisely, we take this homological vanishing property as the *definition* of being Koszul; the discussion of §4 explains why this is the correct definition. In particular, we show that with this definition, if a ring is *Koszul* then it is necessarily *quadratic* (4.10). This also means that we do not need to first construct an admissible basis (or any basis at all) for $\Delta$, as is typical in many proofs of the Koszul property.

To prove that $H_*\overline{\mathcal{B}}(\Delta)[k]$ is concentrated in degree $k$, we look at the combinatorics of the bar construction $\mathcal{B}(\widetilde{C}) = \mathcal{B}(\widetilde{C}, \widetilde{C}, \widetilde{C})$, which are governed by partitions. In particular, there is a weak equivalence of simplicial spaces

$$\mathcal{B}(\widetilde{C})(X) \approx \coprod_{m\geq 0} (P_m \times X^m)_{h\Sigma_m}$$

where $P_m$ is the nerve of the poset of partitions of an $m$-element set (6.2). (The simplicial coordinate comes from the simplicial set $P_m$.) Translating this into a statement about $\overline{\mathcal{B}}(\Delta)$, we discover (7.14) that $\overline{\mathcal{B}}(\Delta)[k]$ is isomorphic to the cokernel of a certain map

$$E_*^\wedge(\overline{P}_m \wedge (S^0 \vee S^0)^{\wedge m})_{h\Sigma_m} \to E_*^\wedge(\overline{P}_m \wedge (S^0)^{\wedge m})_{h\Sigma_m}$$

where $m = p^k$, and $\overline{P}_m$ is certain quotient of $P_m$. This cokernel is denoted $\widetilde{Q}(\overline{P}_m)$ in the text, where $\widetilde{Q}(Y)$ is called the *transitive $E$-homology* (7.1) of a $\Sigma_m$-space $Y$. That is, $\widetilde{Q}(Y)$ is the linearization of the functor $X \mapsto E_*^\wedge(\Sigma_+^\infty Y \wedge X^{\wedge m})_{h\Sigma_m}$ evaluated at $X = S^0$.

Thus, the proof is reduced to showing that the simplicial abelian group $\widetilde{Q}(\overline{P}_m)$ has its homology concentrated in degree $k$. We observe (8.4) that the homology of this simplicial abelian group is precisely the *Bredon homology* of the reduced partition complex with coefficients in the Mackey functor defined by $Q$:

$$H_*(\widetilde{Q}(\overline{P}_m)) \approx H_*^{\Sigma_m}(\overline{P}_m; Q).$$

The result follows via an application (8.3) of a theorem of Arone, Dwyer, and Lesh [ADL16].

1.10. **Historical remarks.** The classical example of an algebra of power operations which is Koszul is the May-Dyer-Lashof algebra of power operations in the homology of a differential graded $E_\infty$-algebra over $\mathbb{F}_p$. That this algebra is Koszul (with respect to an appropriately chosen grading) appears to be well-known, though I don't know an explicit reference; it is implicitly proved in [AM99, §3.1]. The proof for the prime 2 is an application of the PBW basis theorem of Priddy; an adjustment needs to be made to give a proof at odd primes. Kuhn has an elegant unpublished proof that this algebra is Koszul (at least at the prime 2) which bypasses the need to find an admissible basis.

Power operations were first constructed for Morava $E$-theory by Ando [And95], using power operations for $MU$ and suitable choices of orientations. Soon after, Hopkins and Miller were able to show that Morava $E$-theories are $E_\infty$-ring spectra, although it took a long time for the technical details to be worked out. Further work by Ando, Hopkins, and Strickland allows for a description of the ring $\Gamma$ of power operations, in terms of the relevant part of the $E$-cohomology of symmetric groups. The key result here is Strickland's identification of a quotient of $E^*B\Sigma_{p^r}$ as the ring classifying subgroups of a formal group [Str97], [Str98]. Some exposition of these results is given in [Rez09].

The ring of power operations for height 1 Morava $E$-theories amounts to the case of $p$-adic $K$-theory; this case is understood by work of McClure [McC83]. The first (partial) calculation of a power operations algebra for height 2 was carried out by Kashiwabara [Kas95]. What he really did is find a basis for the Morava $K(2)$-homology of symmetric groups. In our language he did this by understanding $\Gamma/(p, v_1)\Gamma$. The ideal $(p, v_1)\Gamma \subset \Gamma$ is not a two-sided ideal, so $\Gamma/(p, v_1)\Gamma$ is not actually a ring. Thus Kashiwabara (aware of this) only computed a product up an indeterminacy. His calculations nonetheless make clear that at height 2, the ring $\Gamma$ is a quadratic algebra, and that $\Gamma$ has an "admissible basis" in terms of certain monomials in the generators, and that the algebra should satisfy the PBW condition of Priddy; thus $\Gamma$ is Koszul at height 2.

Ando, Hopkins, and Strickland conjectured that there is a small resolution (of length $n$) for modules over the ring of power operations of a height $n$ Morava $E$-theory, for any $n$; that is, that these rings are Koszul. Moreover, they explicitly describe a model for this resolution; the description involves the "building complex" of the finite subgroup schemes of a formal group. A brief discussion of these ideas are given in [Str97, §14]. We do not address their "building complex" construction in this paper; however, in [Rez12] we have described a version of the building complex for height $n = 2$, using elliptic curves.

I announced the theorem of this paper in a talk in Mainz in 2005. I later realized that the proof I believed I had at that time was not complete; I found a correct proof in 2008, which was posted to the arXiv in 2012. In 2012 Kathryn Lesh pointed out to me that her work with Arone and Dwyer [ADL16] led to a significant simplification of the argument, which is incorporated in this version.

1.11. **Acknowledgments.** The ideas in the proof given here owe a great deal to the work of Arone and Mahowald on the Goodwillie tower of the identity functor [AM99]. I am of course indebted to the work of Ando, Hopkins, and Strickland on power operations for Morava $E$-theory, which is the foundation of the present work. Finally, I would like to thank Greg Arone, Bill Dwyer, and Kathryn Lesh for sharing their work on the Bredon homology of the partition complex. Finally, I'd like the many people who over the years have personally educated me about the algebraic theory of power operations, a list which includes (but is not limited to): Mike Hopkins, Neil Strickland, Matt Ando, and Paul Goerss.

The author was supported under NSF grants DMS-0203936, DMS-0505056, and DMS-1006054 during the course of this project.

## 2. Monads and bar constructions

In this section we set up some properties of exponential monads, of which the key example is the algebraic approximation functor $\mathbb{T}$ which governs power operations of Morava $E$-theory, to be discussed in §3. In particular, we describe how a suitable grading of an exponential monad determines a grading of its bar-complex.

2.1. **Exponential monads.** Let $\mathcal{C}$ be a symmetric monoidal category with monoidal product $\otimes$ and unit $\mathbb{1}$, and suppose also that $\mathcal{C}$ admits finite coproducts (denoted "$\oplus$", with initial object 0), and that $\otimes$ distributes over coproducts. For convenience, we also assume that inclusions of direct summands are always monomorphisms in $\mathcal{C}$. By an **exponential monad**, we mean a monad equipped with natural isomorphisms

$$\upsilon\colon \mathbb{1} \to T(0), \qquad \zeta\colon T(X) \otimes T(Y) \to T(X \oplus Y),$$

where the map $\zeta$ is a natural transformation of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, with the property that $(\upsilon, \zeta)$ make $T\colon \mathcal{C}^{\oplus} \to \mathcal{C}^{\otimes}$ into a strong symmetric monoidal functor. Furthermore, we require that every $T$-algebra $(A, \phi\colon TA \to A)$ is naturally a commutative monoid object in the symmetric monoidal category $\mathcal{C}$, with unit $\mathbb{1} \xrightarrow{\upsilon} T(0) \xrightarrow{T(0)} T(A) \xrightarrow{\psi} A$ and product $A \otimes A \xrightarrow{\eta\otimes\eta} TA \otimes TA \xrightarrow{\zeta} T(A \oplus A) \xrightarrow{T(\nabla)} TA \xrightarrow{\psi} A$.

The canonical example of an exponential monad is the free commutative algebra monad on the category of abelian groups. The examples we need to work with will be free $E_\infty$-algebra monads on some homotopy category of spaces or spectra, or monads derived from such.

2.2. **Graded exponential monads.** By a **graded exponential monad**, we mean an exponential monoidal monad $T$, together with functors $T\langle m\rangle\colon \mathcal{C} \to \mathcal{C}$ and natural monomorphisms $\gamma_m\colon T\langle m\rangle(X) \to T(X)$, which fit together to give a direct sum decomposition

$$(\gamma_m)\colon \bigoplus_{m\geq 0} T\langle m\rangle(X) \xrightarrow{\sim} T(X)$$

and such that there exist (necessarily unique, because the $\gamma_k$ are monomorphisms) dotted arrows in

$$\begin{array}{ccc} \mathbb{1} & \overset{\upsilon_0}{\dashrightarrow} & T\langle 0\rangle(0) \\ & \searrow^{\upsilon} & \downarrow{\scriptstyle \gamma_0} \\ & & T(0) \end{array} \qquad \begin{array}{ccc} T\langle p\rangle(X) \otimes T\langle q\rangle(Y) & \overset{\zeta_{p,q}}{\dashrightarrow} & T\langle p+q\rangle(X \oplus Y) \\ {\scriptstyle \gamma_p\otimes\gamma_q}\downarrow & & \downarrow{\scriptstyle \gamma_{p+q}} \\ T(X) \otimes T(Y) & \xrightarrow[\zeta]{} & T(X \oplus Y) \end{array}$$

$$\begin{array}{ccc} X & \overset{\eta_1}{\dashrightarrow} & T\langle 1\rangle(X) \\ & \searrow^{\eta} & \downarrow{\scriptstyle \gamma_1} \\ & & T(X) \end{array} \qquad \begin{array}{ccc} T\langle p\rangle T\langle q\rangle(X) & \overset{\mu_{p,q}}{\dashrightarrow} & T\langle pq\rangle(X) \\ {\scriptstyle \gamma_p\gamma_q}\downarrow & & \downarrow{\scriptstyle \gamma_{pq}} \\ TT(X) & \xrightarrow[\mu]{} & T(X) \end{array}$$

such that $T\langle 0\rangle(0) \to T\langle 0\rangle(X)$ and $\eta_1\colon X \to T\langle 1\rangle(X)$ are isomorphisms for all objects $X$. These conditions imply that

$$\mathbb{1} \to T\langle 0\rangle(X), \qquad (\zeta_{p,q})\colon \bigoplus_{p+q=m} T\langle p\rangle(X) \otimes T\langle q\rangle(Y) \to T\langle p+q\rangle(X \oplus Y)$$

are isomorphisms. Furthermore, each composite $T^{\circ q}$ admits a direct sum decomposition $T^{\circ q} \approx \bigoplus_{m\geq 0} T^{\circ q}\langle m\rangle$, determined inductively by

$$T^{\circ q}\langle m\rangle(X) \approx \bigoplus_{m=\sum_j jm_j} \left[ \bigotimes_{j\geq 0} T\langle m_j\rangle(T^{\circ(q-1)}\langle j\rangle(X)) \right],$$

and this decomposition is compatible with the structure maps of the monad. That is, each map $T^{\circ i}\circ\mu\circ T^{\circ j}\colon T^{\circ(i+j+1)}\to T^{\circ(i+j)}$ restricts to a coproduct of maps $T^{\circ(i+j+1)}\langle m\rangle\to T^{\circ(i+j)}\langle m\rangle$, and similarly for $T^{\circ i}\circ\eta\circ T^{\circ j}$.

We refer to $T^{\circ q}\langle m\rangle$ as the **weight** $m$ part of $T^{\circ q}$. Note that if $m=\prod_{i=1}^q m_i$, then there is an evident map

$$T\langle m_1\rangle\circ\cdots\circ T\langle m_q\rangle\to T^{\circ q}\langle m\rangle,$$

which is an inclusion of a direct summand. We say that summands of this form have **pure weight** $m$.

2.3. **The positive part of a graded exponential monad.** Given a graded exponential monad $T$ on $\mathcal{C}$, we write $\widetilde{T}$ for the subfunctor $\widetilde{T}(X)=\bigoplus_{m\geq 1}T\langle m\rangle(X)$ consisting of the part in positive weight. It is clear that $\widetilde{T}$ is a monad in its own right, so that the inclusion map $\widetilde{T}\to T$ is a map of monads. We can similarly speak of the weight $m$ part $\widetilde{T}^{\circ q}\langle m\rangle$ of $\widetilde{T}^{\circ q}$, which will be a subobject of $T^{\circ q}\langle m\rangle$. (Note that $\widetilde{T}^{\circ q}\langle m\rangle$ will not generally be equal to $T^{\circ q}\langle m\rangle$ if $q\geq 2$, as the latter contains a large number of contributions from the weight 0 part of $T$.)

2.4. **Bar complexes.** Given a monad $(T,\eta\colon I\to T,\mu\colon T\circ T\to T)$ on a category $\mathcal{C}$, we will write $\mathcal{B}(T)=\mathcal{B}(T,T,T)$ for the two-sided bar construction for $T$; this is an augmented simplicial object in endofunctors of $\mathcal{C}$, with $\mathcal{B}_q(T)=T^{\circ(q+2)}$. More generally, given $M,N\colon\mathcal{C}\to\mathcal{C}$ which are left and right modules for $T$ respectively, there is a bar construction $\mathcal{B}(M,T,N)$ with $\mathcal{B}_q(M,T,N)\approx M\circ T^{\circ q}\circ N$.

Now suppose $T$ is a graded exponential monad. For each $q\geq 0$ and $m\geq 0$ we define

$$\mathcal{B}_q(T)\langle m\rangle\overset{\text{def}}{=}T^{\circ(q+2)}\langle m\rangle,$$

using the direct sum decomposition $T^{\circ(q+2)}\approx\bigoplus_{m\geq 0}T^{\circ(q+2)}\langle m\rangle$ described above. Thus, the simplicial endofunctor $\mathcal{B}(T)\approx\bigoplus_{m\geq 0}\mathcal{B}(T)\langle m\rangle$ admits a weight decomposition.

We may similarly consider the bar construction of the positive part $\widetilde{T}$, and we similarly obtain a weight decomposition $\mathcal{B}(\widetilde{T})\approx\bigoplus_{m\geq 1}\mathcal{B}(\widetilde{T})$.

2.5. *Example.* Let $O$ be an $E_\infty$-operad in spaces, and let $C$ denote the monad on spaces defined by $C(X)\overset{\text{def}}{=}\coprod_{m\geq 0}(O(m)\times X^m)_{h\Sigma_m}$. The functor $C$ descends to a monad on the homotopy category $h$Spaces of spaces, which we also denote by $C$. This is a graded exponential monad in our sense; the graded pieces are $C\langle m\rangle(X)\approx(O(m)\times X^m)_{h\Sigma_m}$.

The corresponding bar complex admits a grading $\mathcal{B}(C)\approx\coprod_{m\geq 0}\mathcal{B}(C)\langle m\rangle$; applied to a space, we obtain a decomposition $\mathcal{B}(C)(X)\approx\coprod_{m\geq 0}\mathcal{B}(C)\langle m\rangle(X)$ of simplicial spaces. As we will note below (7.10), for the positive part of $C$ there is a natural weak equivalence

$$\mathcal{B}(\widetilde{C})\langle m\rangle(X)\approx(P_m\times X^m)_{h\Sigma_m}$$

of simplicial spaces, where $P_m$ is the partition complex (6.2) on the set of $m$ elements.

2.6. *Example.* The monad $D$ on the homotopy category of spectra, defined by $D(Y)\overset{\text{def}}{=}\bigvee_{m\geq 0}(O(m)_+\wedge Y^{\wedge m})_{h\Sigma_m}$, is similarly an exponential monad.

2.7. *Example.* Let $\mathbb{P}$ denote the free commutative $E$-algebra monad, defined on the category $\mathrm{Mod}_E$ of $E$-module spectra. This functor descends to a functor on the homotopy category $h\mathrm{Mod}_E$, which we also denote by $\mathbb{P}$. As such, it is a graded exponential functor, with $\mathbb{P} \approx \bigvee_{m\geq 0} \mathbb{P}\langle m\rangle$, where $\mathbb{P}\langle m\rangle(M) \approx (M^{\wedge_E m})_{h\Sigma_m}$. We will typically write $\mathbb{P}_m$ for $\mathbb{P}\langle m\rangle$.

2.8. **Associations.** Let $T$ and $T'$ be graded exponential monads on suitable categories $\mathcal{C}$ and $\mathcal{C}'$. An **association** from $T$ to $T'$ is a functor $G\colon \mathcal{C} \to \mathcal{C}'$ which is equipped with the structure of a weak monoidal functor in two different ways, namely as functors $C_\oplus \to C'_\oplus$ and $C_\otimes \to C'_\otimes$, together with a natural map $TG \to GT'$ which is compatible with all the structure.

2.9. *Example.* The functor $\Sigma^\infty_+\colon h\mathrm{Spaces} \to h\mathrm{Spectra}$ defines as association between $C$ and $D$. Likewise, the functor $\Sigma\colon h\mathrm{Spectra} \to h\mathrm{Spectra}$ defines an association between $D$ and itself.

2.10. *Example.* There is an association between the monads $C$ and $\mathbb{P}$ described above, given by the functor $E \wedge \Sigma^\infty_+\colon h\mathrm{Spaces} \to h\mathrm{Mod}_E$. In particular: (i) $E \wedge \Sigma^\infty_+$ takes coproducts to coproducts; (ii) $E \wedge \Sigma^\infty_+$ takes products to smash products; (iii) there is a natural map (in fact, a weak equivalence)

$$\mathbb{P}(E \wedge \Sigma^\infty_+ X) \to E \wedge \Sigma^\infty_+ C(X),$$

which is compatible with the exponential structures on the monads, and which is compatible with the gradings, in the sense that it restricts to maps $E \wedge \Sigma^\infty_+ C\langle m\rangle(X) \to \mathbb{P}\langle m\rangle(X)$.

2.11. *Example.* One more example of exponential monad is given in the next section, where we describe a monad $\mathbb{T}$ on the category $\mathrm{Mod}_{E_*}$ of $E_*$-modules. There is an association between $\mathbb{T}$ and the monad $\mathbb{P}$ above, given by $\pi_* L_{K(n)}\colon h\mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$, so that there is a natural map

$$\mathbb{T}(\pi_* L_{K(n)} M) \to \pi_* L_{K(n)} \mathbb{P}(M).$$

## 3. Rings of power operations

The homotopy groups of a $K(n)$-local commutative $E$-algebra spectrum are naturally algebras over a certain monad $\mathbb{T}$, which captures algebraically the $E$-homology of symmetric groups. In this section, we recall from [Rez09] properties of the monad $\mathbb{T}$. From this monad, we will extract two kinds of graded rings of "power operations"; the ring $\Gamma^q$, which is a ring of *additive* operations on $\pi_{-q}$ of a $K(n)$-local commutative $E$-algebra, and the ring $\Delta^q$, which is a ring of operations on the degree $-q$ part of the cotangent space of an augmented algebra. The main result of this section is that all of the rings in question are isomorphic. It is the ring $\Delta \overset{\text{def}}{=} \Delta^0$ which we will explicitly show is Koszul in subsequent sections.

3.1. **Introduction.** Before diving in to the case of Morava $E$-theory, it may be useful to consider things in a more general context.

Fix an arbitrary structured commutative ring spectrum $E$, and the category $\mathrm{Alg}_E$ of commutative $E$-algebra spectra. Consider the problem of determining all *natural operations* on the homotopy groups of commutative $E$-algebras. That is, for all $i, j \in \mathbb{Z}$ we would like to compute the set of natural transformations

$$\pi_i(-) \to \pi_j(-) \quad \text{of functors } h\mathrm{Alg}_E \to \mathrm{Set}.$$

We might instead restrict to the case of *additive* operations, i.e., natural transformations of functors $h\mathrm{Alg}_E \to \mathrm{Ab}$.

Let us consider how this works out in the case that $E = H\mathbb{F}_p$, the mod $p$ Eilenberg MacLane spectrum.[1]

- Commutative $H\mathbb{F}_p$-algebras are algebras for the free symmetric $H\mathbb{F}_p$-algebra monad $\mathbb{P}$ on $\mathrm{Mod}_{H\mathbb{F}_p}$, which as described above (2.7) is an example of an exponential monad.
- This monad descends to an exponential monad on the homotopy category $h\mathrm{Mod}_{H\mathbb{F}_p}$ of modules, which we also denote by $\mathbb{P}$. The underlying object in $h\mathrm{Mod}_{H\mathbb{F}_p}$ of a commutative $H\mathbb{F}_p$-algebra spectrum is naturally an algebra for $\mathbb{P}$ on $h\mathrm{Mod}_{H\mathbb{F}_p}$ (this kind of structure is called an $H_\infty$-$H\mathbb{F}_p$-algebra structure).
- Taking homotopy groups defines a symmetric monoidal equivalence $\pi_*\colon h\mathrm{Mod}_{H\mathbb{F}_p} \xrightarrow{\sim} \mathrm{Mod}^*_{\mathbb{F}_p}$ to the category of graded $\mathbb{F}_p$-vector spaces.
- Thus the monad $\mathbb{P}$ on $h\mathrm{Mod}_{H\mathbb{F}_p}$ descends to a monad $\mathbb{T}$ on $\mathrm{Mod}^*_{\mathbb{F}_P}$, which we regard as an "algebraic approximation" to $\mathbb{P}$. It is a very good approximation indeed, as by construction it comes with a monoidal natural isomorphism
$$\alpha\colon \mathbb{T}(\pi_* M) \xrightarrow{\sim} \pi_*\mathbb{P}(M)$$
of functors $h\mathrm{Mod}_{H\mathbb{F}_P} \to \mathrm{Mod}^*_{\mathbb{F}_P}$, which is an example of an association between the monads $\mathbb{P}$ and $\mathbb{T}$ as defined in (2.8). Furthermore, $\mathbb{T}$ inherits from $\mathbb{P}$ the structure of an exponential monad.
- The object $\mathbb{T}(\mathbb{F}_p[i]) \approx \pi_*\mathbb{P}(\Sigma^i H\mathbb{F}_p)$ is the free $\mathbb{T}$-algebra on one generator in degree $i$. (Here we write $V[i]$ for the shifted graded vector space, with $V[i]_j = V_{-i+j}$.) It is a formal consequence that the set of natural transformations $\pi_i(-) \to \pi_j(-)$ of functors $h\mathrm{Alg}_{H\mathbb{F}_p} \to \mathrm{Set}$ is equal to
$$\pi_j\mathbb{P}(\Sigma^i H\mathbb{F}_p) \approx \big(\mathbb{T}(\mathbb{F}_p[i])\big)_j,$$
the degree $j$ part of the free algebra on a generator in degree $i$.
- Addive operations, i.e., natural transformations $\pi_i(-) \to \pi_j(-)$ of functors $h\mathrm{Alg}_{H\mathbb{F}_p} \to \mathrm{Ab}$, correspond precisely to the *primitive* elements in the free algebra, i.e., elements in the equalizer of the pair of maps
$$\mathbb{T}(i_1+i_2), \mathbb{T}(i_1)+\mathbb{T}(i_2)\colon \mathbb{T}(\mathbb{F}_p[i])_j \rightrightarrows \mathbb{T}(\mathbb{F}_p[i] \oplus \mathbb{F}_p[i])_j,$$
where $i_1, i_2\colon \mathbb{F}_p[i] \to \mathbb{F}_p[i] \oplus \mathbb{F}_p[i]$ are the evident inclusion maps.

Thus, the theory of homotopy operations for commutative $H\mathbb{F}_p$-algebra spectra is entirely described by an essentially algebraic object, the monad $\mathbb{T}$ on graded vector spaces. The structure of $\mathbb{T}$ and its category of algebras is entirely understood: it is the category of graded $\mathbb{F}_p$-algebras equipped with *Dyer-Lashof operations* $\{Q^s, \beta Q^s\}$, which satisfy an explicit set of axioms[2].

[1]This is inspired by the treatments in [May70] and in [BMMS86] (especially § IX.2), though not identical to what those authors write, since after all the foundations of structured module spectra were not available at that time.

[2]Essentially [BMMS86, Ch. III, Thm. 1.1, (1)–(7)], with the modification that in the homotopy of commutative $H\mathbb{F}_p$-algebras there is no Bockstein operation $\beta$; the Bockstein is defined only on $H\mathbb{F}_p$-algebras of the form $H\mathbb{F}_p \wedge_S A$ for commutative $S$-algebras $A$. Rather, one posits additional operations "$\beta Q^s$" which satisfy suitable identities. See [May70] for a concrete description of this structure.

3.2. *Remark.* This story goes much the same way for algebras over the rational Eilenberg-MacLane spectrum $H\mathbb{Q}$. In this case, the monad $\mathbb{T}$ on the category of graded rational vector spaces turns out to be identical to the usual (graded) symmetric algebra monad.

We can carry out a variant of the scenario in which $H\mathbb{F}_p$ is replaced by a Morava $E$-theory spectrum. This was done in [Rez09], and we will summarize the main points we need below. Before doing so, we note some of the points in the above outline which require modification.

- The homotopy category $h\mathrm{Mod}_E$ of $E$-modules is not equivalent to the category of graded $E_* = \pi_* E$-modules. However, there is a full subcategory $h\mathrm{Mod}_E^{\mathrm{free}} \subset h\mathrm{Mod}_E$ of "free" modules (i.e., those for which $\pi_* M$ is a free $E_*$-module), which is equivalent to the full subcategory $\mathrm{Mod}_{E_*}^{\mathrm{free}} \subset \mathrm{Mod}_{E_*}$ of free graded $E_*$-modules.
- For a free $E$-module $M \approx \bigvee \Sigma^{i_\alpha} E$, it is rarely the case that $\mathbb{P}(M) \approx \bigvee_m M_{h\Sigma_m}^{\wedge_E m}$ is free. This is not specific to Morava $E$-theory: for instance, $E_{h\Sigma_m}^{\wedge_E m} \approx E \wedge \Sigma_+^\infty B\Sigma_m \approx E \vee (E \wedge \Sigma^\infty B\Sigma_m)$, and $\pi_*(E \wedge \Sigma^\infty B\Sigma_m)$ is torsion for any homology theory $E$.
- However, it is true [Rez09, Prop. 3.17] that for a *finitely generated* free $E$-module $M$, the spectra $L_{K(n)}\mathbb{P}_m(M) \approx L_{K(n)}(M_{h\Sigma_m}^{\wedge_E m})$ obtained by taking the $K(n)$-localization of the weight-homogenous pieces of $\mathbb{P}(M)$ are also finitely generated and free.
- The $K(n)$-localization functor $L_{K(n)}$ does not preverve coproducts of spectra. Thus, $L_{K(n)}\mathbb{P}(M)$ is not generally a free $E$-module if $M$ is free. What is true in this case is that $\pi_* L_{K(n)}\mathbb{P}(M)$ is the $\mathfrak{m}$-adic completion of a free $E_*$-module, where $\subset E_0$ is the maximal ideal of the coefficient ring.

For the purposes of getting at rings of additive operations on $K(n)$-local commutative $E$-algebras, it suffices to deal with $L_{K(n)}\mathbb{P}_m(M)$ for finite free $E$-modules $M$. Thus we will define "algebraic approximation" functors $\mathbb{T}_m\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ by a Kan extension of $\pi_* L_{K(n)}\mathbb{P}_m$ restricted to finite free $E$-modules. We will also construct a natural "approximation map"

$$\alpha\colon \mathbb{T}\pi_*(-) = \bigoplus_m \mathbb{T}_m \pi_*(-) \to \pi_* L_{K(n)}\mathbb{P}(-)$$

of functors $h\mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$, which is an isomorphism for finite free $E$-modules.

3.3. *Remark.* It is important to note that $\alpha$ is *not* generally an isomorphism, even on finite free $E$-modules. What is true is that for a free (but not necessarily finite) $E$-module $M$, the induced map

$$(\mathbb{T}\pi_* M)_{\mathfrak{m}}^{\wedge} \xrightarrow{\sim} \pi_* L_{K(n)}\mathbb{P}(M)$$

from completion with respect to the maximal ideal $\mathfrak{m} \subset E_0$ is an isomorphism. We address this issue and its relevance to operations in (3.22).

Another approach to this issue is to construct a "better" approximation functor, defined on a suitable category of *completed* $E_*$-modules. Such an approximation functor comes with an approximation map which is an isomorphism on all (completed) free $E$-modules. Such a better approximation functor is constructed by Barthel and Frankland [BF15]. This is a useful construction for the sake of applications, but does not aid in proving the theorem of this paper, so we will not make use of it.

3.4. **Completed $E$-homology.** We now fix a Morava $E$-theory spectrum, with height $n$.

Given a spectrum $Y$, we define the **completed $E$-homology** of $Y$ by

$$E_*^\wedge Y \overset{\text{def}}{=} \pi_* L_{K(n)}(E \wedge Y).$$

If $X$ is a space, we write $E_*^\wedge X$ for $E_*^\wedge \Sigma_+^\infty X$. As is well-known, the functor $E_*^\wedge$ satisfies the Eilenberg-Steenrod axioms but not Milnor's wedge axiom.

3.5. **Algebraic structure on the homotopy groups of $E$-algebras.** As we noted above (2.7), the free commutative $E$-algebra functor $\mathbb{P}\colon h\mathrm{Mod}_E \to h\mathrm{Mod}_E$ carries the structure of a graded exponential monad with respect to derived smash product of $E$-modules; in particular, it admits a decomposition $\mathbb{P} \approx \bigvee_{m\geq 0} \mathbb{P}_m$. We need one more piece of structure; namely, for all $m \geq 1$, there is a natural transformation $e\colon \Sigma\mathbb{P}_m \to \mathbb{P}_m\Sigma$, defined because $\mathbb{P}_m$ is compatible with the enrichment of $\mathrm{Mod}_E$ over pointed spaces.

Following [Rez09, §4.4] we define the **algebraic approximation functor** $\mathbb{T}_m\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ as a left Kan extension

$$\mathbb{T}_m := \mathrm{LKan}_{\pi_* i}\, \pi_* L_{K(n)} \mathbb{P}_m i$$

with respect to the functors in the diagram

$$\begin{array}{ccccc} h\mathrm{Mod}_E^{\text{finite free}} & \xrightarrow{\;\;i\;\;} & h\mathrm{Mod}_E & \xrightarrow{\pi_* L_{K(n)}\mathbb{P}_m} & \mathrm{Mod}_{E_*} \\ {\scriptstyle \pi_*}\downarrow{\scriptstyle \sim} & & {\scriptstyle \pi_*}\downarrow & \nearrow{\scriptstyle \mathbb{T}_m} & \\ \mathrm{Mod}_{E_*}^{\text{finite free}} & \xrightarrow[\;\;j\;\;]{} & \mathrm{Mod}_{E_*} & & \end{array}$$

Here $i$ and $j$ are the evident fully faithful inclusions. By construction (since $\pi_* i \approx j\pi_*$ is fully faithful) there is a natural isomorphism $\mathbb{T}_m \pi_* i \approx \pi_* L_{K(n)} \mathbb{P}_m i$, i.e., $\mathbb{T}_m(\pi_* M)$ computes the homotopy groups of $L_{K(n)}\mathbb{P}_m(M)$ when $E$ is a finitely generated free $E$-module. In [Rez09, §4.6] it is shown that this equivalence extends to an **approximation map**

$$\alpha_m\colon \mathbb{T}_m(\pi_* M) \to \pi_* L_{K(n)} \mathbb{P}_m(M),$$

a natural transformation of functors $h\mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$.

We define $\mathbb{T} := \bigoplus_m \mathbb{T}_m\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$, and obtain an approximation map

$$\alpha\colon \mathbb{T}(\pi_* M) \to \pi_* L_{K(n)} \mathbb{P}(M)$$

(using the natural map $\bigvee_m L_{K(n)}\mathbb{P}_m \to L_{K(n)}(\bigvee_m \mathbb{P}_m) = L_{K(n)}\mathbb{P}$).

3.6. **Properties of the algebraic approximation.** We summarize here the salient properties of the of the algebraic approximation.

(1) The functor $\mathbb{T}$ is equipped with the structure of an exponential monad with respect to tensor product of $E_*$-modules [Rez09, Prop. 4.10, §4.13], and the approximation map $\alpha$ [Rez09, §4.6] is an association between $\mathbb{T}$ and $\mathbb{P}$ in the sense of (2.8). (This is clear from the construction of $\gamma_k$ in [Rez09, §4.13].)

(2) The decomposition $\mathbb{T} = \bigoplus_m \mathbb{T}_m$ gives the structure of a *graded* exponential monad, [Rez09, §4.4, §4.13], corresponding to the analogous decomposition of $\mathbb{P}$. We will sometimes write typically we write $\mathbb{T}\langle m\rangle$ for $\mathbb{T}_m$. Furthermore, the approximation map descends to the grading, via the maps $\alpha_m$.

(3) If $M$ is an $E$-module such that $\pi_*M$ is finite and free as an $E_*$-module, then $\alpha_m$ evaluated at $M$ is an isomorphism [Rez09, Prop. 4.8].

(4) The approximation maps $\alpha_m$ give rise to a natural transformation

$$\widehat{\alpha}\colon L_0\mathbb{T}(\pi_*M) \to \pi_*L_{K(n)}\mathbb{P}(M)$$

which is an isomorphism when $\pi_*M$ is a flat $E_*$-module [Rez09, Prop. 4.17]. Here $L_0$ denotes the 0th left derived functor of the $\mathfrak{m}$-adic completion functor $(-)^\wedge_{\mathfrak{m}}\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$, where $\mathfrak{m} \subset \pi_0E$ is the maximal ideal.

We note that if $M_*$ is a free $E_*$-module, then $\mathbb{T}(M_*)$ is also free, whence $L_0\mathbb{T}(M_*)$ is isomorphic to the $\mathfrak{m}$-adic completion of the free module $\mathbb{T}(M_*)$. Thus, if $\pi_*M$ is a free $E_*$ module, then we get an isomorphism $\mathbb{T}(\pi_*M)^\wedge_{\mathfrak{m}} \xrightarrow{\sim} \pi_*L_{K(n)}\mathbb{P}(M)$.

(5) If $N$ is a finite and free $E_*$-module, then so is $\mathbb{T}_mN$ for all $m \geq 0$ (by (3) and [Rez09, Prop. 3.16]). Thus for all $q \geq 0$, the approximation maps induce isomorphisms

$$\mathbb{T}^{\circ q}\langle m\rangle(\pi_*M) \xrightarrow{\sim} \pi_*L_{K(n)}(\mathbb{P}^{\circ q}\langle m\rangle M)$$

when $\pi_*M$ is finite and free.

(6) The functor $\mathbb{T}$ and each functor $\mathbb{T}_m$ commutes with filtered colimits and reflexive coequalizers [Rez09, Prop. 4.12].

(7) If $M$ is an $E_*$-module concentrated in even degree, then so is $\mathbb{T}M$. (This follows using the Kan extension property (5), the isomorphism (3), and the fact that $E^\wedge_*B\Sigma_m$ are concentrated in even degree.)

We write $\mathrm{Alg}^*_{\mathbb{T}}$ for the category of algebras for the monad $\mathbb{T}$ on $\mathrm{Mod}_{E_*}$.

(8) The category $\mathrm{Alg}^*_{\mathbb{T}}$ is complete and cocomplete; limits are computed in the underlying category $\mathrm{Mod}_{E_*}$, and colimits are computed in the underlying category $\mathrm{Alg}_{E_*}$ of commutative $E_*$-algebras. In particular, coproducts in $\mathrm{Alg}^*_{\mathbb{T}}$ are tensor products of $E_*$-modules [Rez09, Cor. 4.19].

(9) Using the approximation map, we see that every algebra $A$ for the monad $\mathbb{P}$ on $h\mathrm{Mod}_E$ (i.e., for every $H_\infty$-$E$-algebra), the homotopy groups $\pi_*L_{K(n)}A$ naturally carry the structure of a $\mathbb{T}$-algebra. That is, we obtain a functor

$$\pi_*L_{K(n)}\colon \mathrm{Alg}_{\mathbb{P}} \to \mathrm{Alg}^*_{\mathbb{T}}$$

lifting $\pi_*L_{K(n)}\colon \mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$.

(10) Let $\Sigma\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ denote the functor $\Sigma M \overset{\mathrm{def}}{=} E_*S^1 \otimes_{E_*} M$. For $m \geq 1$, there is an algebraic **suspension map**

$$E\colon \Sigma\mathbb{T}_m \to \mathbb{T}_m\Sigma,$$

which with respect to the approximation map is compatible with the topological suspension map $e\colon \Sigma\mathbb{P}_m \to \mathbb{P}_m\Sigma$. Furthermore, the algebraic suspension map is compatible with the monad structure on $\mathbb{T}$, in the sense that both ways of building a map $\Sigma(\mathbb{T}\mathbb{T})\langle m\rangle \to (\mathbb{T}\langle m\rangle)\Sigma$ coincide. (This suspension map is denoted $E_1$ in [Rez09, §4.25], where the notation $\omega^{-q/2}$ is used for $E_*S^q$. Compatibility between algebraic and topological suspension maps is by construction. The algebraic suspension map is defined for finite free $E_*$-modules by means of the approximation isomorphism and the topological suspension map; the general algebraic suspension map is then defined using (5) above. The compatibility with the monad structure is [Rez09, Prop. 4.27].)

We need one more fact about the algebraic suspension: it "kills decomposable elements".

3.7. **Proposition.** *For $i, j \geq 1$, the composition*

$$\Sigma(\mathbb{T}_i M \otimes \mathbb{T}_j N) \xrightarrow{\Sigma\zeta} \Sigma\mathbb{T}_{i+j}(M \oplus N) \xrightarrow{E} \mathbb{T}_{i+j}\Sigma(M \oplus N)$$

*is equal to* 0.

*Proof.* Consider functors $\mathrm{Mod}_E \times \mathrm{Mod}_E \to \mathrm{Mod}_E$ defined by $(X, Y) \mapsto \mathbb{P}_i X \wedge_E \mathbb{P}_j Y$ and $(X, Y) \mapsto \mathbb{P}_{i+j}(X \vee Y)$. These are each enriched over pointed spaces, and thus we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma(\mathbb{P}_i X \wedge \mathbb{P}_j Y) & \longrightarrow & \mathbb{P}_i \Sigma X \wedge \mathbb{P}_j \Sigma Y \\ {\scriptstyle \Sigma\zeta}\downarrow & & \downarrow \\ \Sigma\mathbb{P}_{i+j}(X \vee Y) & \xrightarrow[e]{} & \mathbb{P}_{i+j}(\Sigma X \vee \Sigma Y) \end{array}$$

where the vertical maps come from the exponential structure. We see that the bottom horizontal map is the suspension map $e$, while the top horizontal map factors $\Sigma(\mathbb{P}_i X \wedge \mathbb{P}_j Y) \to \Sigma\mathbb{P}_i X \wedge \Sigma\mathbb{P}_j Y \to \mathbb{P}_i \Sigma X \wedge \mathbb{P}_j \Sigma Y$. The first of these two maps is null, as it uses the diagonal embedding $S^1 \to S^1 \wedge S^1$. Thus the composite $e \circ \zeta$ is null. The corresponding vanishing result for the algebraic suspension map is immediate for finite free modules, and follows in general since the algebraic approximation functors are left Kan extended from finite free modules. □

3.8. **Augmented rings.** Let $\mathrm{Alg}^*_{\mathbb{T}/E_*}$ denote the category of $\mathbb{T}$-objects augmented over $E_*$; an object of $\mathrm{Alg}^*_{\mathbb{T}/E_*}$ is a morphism $A \to E_*$.

We will write $\widetilde{\mathbb{T}}$ for the subfunctor of $\mathbb{T}$ defined by

$$\widetilde{\mathbb{T}}(M) \stackrel{\mathrm{def}}{=} \bigoplus_{m \geq 1} \mathbb{T}_m(M).$$

This is precisely the kernel of the natural augmentation map $\mathbb{T}(M) \to \mathbb{T}(0) = E_*$.

As noted above ((2.3), using (3.6)(2)), the functor $\widetilde{\mathbb{T}}$ itself inherits the structure of a monad on $\mathrm{Mod}_{E_*}$. The category of algebras over $\widetilde{\mathbb{T}}$ is equivalent to the category of $\mathrm{Alg}^*_{\mathbb{T}/E_*}$ of augmented $\mathbb{T}$-algebras. This is a standard observation, so we won't spell out the details, except to note that if $A \to E_*$ is an augmented $\mathbb{T}$-algebra with structure map $\psi\colon \mathbb{T}A \to A$ and augmentation ideal $\widetilde{A}$, then the corresponding $\widetilde{\mathbb{T}}$-algebra structure $\widetilde{\psi}\colon \widetilde{\mathbb{T}}\widetilde{A} \to \widetilde{A}$ is simply the restriction of $\psi$ to $\widetilde{\mathbb{T}}\widetilde{A} \subset \mathbb{T}A$.

3.9. **Abelian group objects.** Recall that the notion of **abelian group object** can be defined in any category with finite products.

3.10. **Proposition.** *An object $A \to E_*$ of $\mathrm{Alg}^*_{\mathbb{T}/E_*}$ with augmentation ideal $\widetilde{A}$ admits the structure of an abelian group object if and only if $\widetilde{A}^2 = 0$, in which case the abelian group structure is unique.*

*Proof.* An abelian group structure is a map $f\colon A\times_{E_*}A\to A$ of $\mathbb{T}$-algebras, which satisfies the axioms for an abelian group; the unit of the abelian group is necessarily given by the unique $\mathbb{T}$-algebra map $E_*\to A$. Since $A\approx E_*\oplus\widetilde{A}$ as $E_*$-modules, we see that the evident $\mathbb{T}$-algebra map $A\otimes_{E_*}A\to A\times_{E_*}A$ is surjective, with kernel $\widetilde{A}\otimes_{E_*}\widetilde{A}$. Thus, a map $f$ satisfying the unit axiom for an abelian group exists if and only if the multiplication map $A\otimes_{E_*}A\to A$ sends $\widetilde{A}\otimes_{E_*}\widetilde{A}$ to 0. That is, a unital $f$ exists if and only if $\widetilde{A}^2=0$. It is straightforward to show that if such a unital $f$ exists, it is given by $f(c,x,y)=c+x+y$ (written in terms of the $E_*$-module decompositions $A\approx E_*\oplus\widetilde{A}$ and $A\times_{E_*}A\approx E_*\oplus\widetilde{A}\oplus\widetilde{A}$), and therefore is the unique abelian group structure on $A$. □

Thus, the category $(\mathrm{Alg}^*_{\mathbb{T}/E_*})_{\mathrm{ab}}$ of abelian group objects is identified with the full subcategory $\mathrm{Alg}^*_{\mathbb{T}/E_*}$ of augmented $\mathbb{T}$-algebras with $\widetilde{A}^2=0$, and the left adjoint to this inclusion is given by $A\mapsto A/\widetilde{A}^2$, which can be regarded as providing the **cotangent space** to $A$ at the augmentation $A\to E_*$.

3.11. **Suspension and loop.** The homotopy category $h(\mathrm{Alg}_{E/E})$ of augmented commutative $E$-algebra spectra admits a **loop** functor $\Omega\colon h(\mathrm{Alg}_{E/E})\to h(\mathrm{Alg}_{E/E})$. If $A$ is an augmented commutative $E$-algebra, then $\Omega A$ is the homotopy pullback

$$\begin{array}{ccc} \Omega A & \longrightarrow & E \\ \downarrow & & \downarrow \\ E & \longrightarrow & A \end{array}$$

in $h(\mathrm{Alg}_{E/E})$. The underlying $E$-module spectrum of $\Omega A$ has the form $E\vee\Sigma^{-1}\widetilde{A}$, where $\widetilde{A}$ is the homotopy fiber of the augmentation $A\to E$.

There is a corresponding loop functor $\Omega\colon \mathrm{Alg}^*_{\mathbb{T}/E_*}\to\mathrm{Alg}^*_{\mathbb{T}/E_*}$, with the property that as an $E_*$-module $\Omega A\approx E_*\oplus(E_*S^{-1}\otimes_{E_*}\widetilde{A})$, where $\widetilde{A}$ is the augmentation ideal. Furthermore, the augmentation ideal of $\Omega A$ will be square-zero, and so $\Omega$ factors through a functor $\mathrm{Alg}^*_{\mathbb{T}/E_*}\to(\mathrm{Alg}^*_{\mathbb{T}/E_*})_{\mathrm{ab}}$.

To define the algebraic loop functor, recall the suspension map $E\colon\Sigma\widetilde{\mathbb{T}}\to\widetilde{\mathbb{T}}\Sigma$ of §3.5(10). Since $\Sigma\colon\mathrm{Mod}_{E_*}\to\mathrm{Mod}_{E_*}$ is a self equivalence, it has an inverse functor $\Sigma^{-1}$, and we may therefore use it to define the **desuspension map** $E'\colon\widetilde{\mathbb{T}}\Sigma^{-1}\to\Sigma^{-1}\widetilde{\mathbb{T}}$.

Let $A\to E_*$ in $\mathrm{Alg}^*_{\mathbb{T}/E_*}$ with augmentation ideal $\widetilde{A}$. As noted in §3.8, it is equivalent to consider $\widetilde{A}$ as an algebra over $\widetilde{\mathbb{T}}$, with structure map $\tilde{\psi}\colon\widetilde{\mathbb{T}}\widetilde{A}\to\widetilde{A}$. We thus *define* $\Omega A$ to be the augmented $\mathbb{T}$-algebra with underlying augmentation ideal $\Sigma^{-1}\widetilde{A}$, and with structure map $\tilde{\psi}_\Omega$ defined as the composite

$$\widetilde{\mathbb{T}}\Sigma^{-1}\widetilde{A}\xrightarrow{E'}\Sigma^{-1}\widetilde{\mathbb{T}}\widetilde{A}\xrightarrow{\Sigma^{-1}\tilde{\psi}}\widetilde{A}.$$

To show that $\tilde{\psi}_\Omega$ defines a $\widetilde{\mathbb{T}}$-algebra structure is a straightforward calculation, which relies on the compatibility of suspension with the monad structure of $\mathbb{T}$ described in 3.5(10). That the augmentation ideal of $\Omega A$ is square-zero amounts to (3.7).

3.12. **The rings $\Gamma^q$.** The ring $\Gamma^q$ is defined to be a ring which naturally acts on the degree $(-q)$-part of the underlying $E_0$-module of a $\mathbb{T}$-algebra, and hence acts naturally on $\pi_{-q}$ of a $K(n)$-local commutative $E$-algebra.

Let $U^q\colon \mathrm{Alg}^*_{\mathbb{T}} \to \mathrm{Ab}$ denote the functor which sends a $\mathbb{T}$-algebra $A$ to its $(-q)$-th grading $A_q$, viewed as an abelian group. We *define* $\Gamma^q \overset{\mathrm{def}}{=} \mathrm{End}(U^q)$, the endomorphism ring of the functor $U^q$; thus, an element $f \in \Gamma^q$ gives a natural abelian group homomorphism $A_{-q} \to A_{-q}$ for all $A$ in $\mathrm{Alg}^*_{\mathbb{T}}$.

The underlying set of $U^q(A)$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Alg}^*_{\mathbb{T}}}(\mathbb{T}(E_*S^{-q}), A)$, where $\mathbb{T}(M)$ represents the free $\mathbb{T}$-algebra on an $E_*$-module $M$. Therefore, we see that endomorphisms of the composite functor $\mathrm{Alg}^*_{\mathbb{T}} \xrightarrow{U^q} \mathrm{Ab} \to \mathrm{Set}$ are exactly the $\mathbb{T}$-algebra endomorphisms of $\mathbb{T}(E_*S^{-q})$. Hence, the monoid of set-endomorphisms of $U^q$ is

$$\mathrm{Hom}_{\mathrm{Alg}^*_{\mathbb{T}}}(\mathbb{T}(E_*S^{-q}), \mathbb{T}(E_*S^{-q})) \approx \mathrm{Hom}_{\mathrm{Mod}_{E_0}}(E_*S^{-q}, \mathbb{T}(E_*S^{-q})) \approx \mathbb{T}(E_*S^{-q})_{-q},$$

and so $\Gamma^q$ may be identified with the degree $-q$ *primitives* of $\mathbb{T}(E_*S^{-q})$. We thus recover the definition of $\Gamma^q$ given in [Rez09, §7], where the notation $\omega^{q/2}$ is used for the $E_*$-module $E_*S^{-q}$.

Using the isomorphisms

$$\mathbb{T}_m(E_*S^{-q}) \approx \pi_* L_{K(n)}\mathbb{P}_m(E \wedge S^{-q}) \approx E^\wedge_* B\Sigma_m^{-q\rho_m}$$

and

$$\mathbb{T}_m(E_*S^{-q} \oplus E_*S^{-q}) \approx \pi_* L_{K(n)}\mathbb{P}_m(E \wedge (S^{-q} \vee S^{-q})) \approx \bigoplus_{0\le j\le m} E^\wedge_* B(\Sigma_j \times \Sigma_{m-j})^{-q\rho_m},$$

we find that

$$\Gamma^q \approx \bigoplus_{k\ge 0} \Gamma^q[k],$$

where

$$\begin{aligned}\Gamma^q[k] &\approx \mathrm{Ker}\left[\mathbb{T}_{p^k}(E_*S^{-q})_{-q} \to \mathbb{T}_{p^k}(E_*S^{-q} \oplus E_*S^{-q})_{-q}\right] \\ &\approx \mathrm{Ker}\left[E^\wedge_{-q} B\Sigma_{p^k}^{-q\rho_{p^k}} \to \bigoplus_{0<j<p^k} E^\wedge_{-q} B(\Sigma_j \times \Sigma_{p^k-j})^{-q\rho_{p^k}}\right],\end{aligned}$$

the map in the second line being induced by the transfer map associated to the inclusion $\Sigma_j \times \Sigma_{p^k-j} \subset \Sigma_{p^k}$.

Note that $E_0 = \Gamma^q[1] \subseteq \Gamma^q$ is a subring, but is not central; this is because elements of $\Gamma^q$ given rise to natural maps of abelian groups, but not necessarily to $E_0$-module maps. Thus each $\Gamma^q[k]$ is naturally an $E_0$-bimodule. The *left* $E_0$-module structure on $\Gamma^q[k]$ coincides with the module structure inherited by the inclusion $\Gamma^q[k] \subseteq E^\wedge_{-q} B\Sigma_{p^k}^{-q\rho_{p^k}}$.

The following is crucial for understanding the structure of $\Gamma^q$. It is proved as [Rez09, Prop. 7.3], though it ultimately derives from [Str98].

3.13. **Proposition.** *Each $\Gamma^q[k]$ is a finitely generated free left $E_0$-module.*

*Proof.* That these are finitely generated and free (and in fact, $\Gamma^q[k]$ is a direct summand of $E^\wedge_{-q}B\Sigma_{p^k}^{-q\rho_{p^k}}$) is proved as [Rez09, Prop. 6.1 and Prop. 7.2], an argument which ultimiately derives from [Str98]. □

We also know the ranks of the $\Gamma^q[k]$; see (3.21) below.

Here is a variant of the above description of $\Gamma^q$, which we will need below. Let $I^q\colon \mathrm{Alg}^*_{\mathbb{T}/E_*} \to \mathrm{Ab}$ denote the functor which sends an augmented $\mathbb{T}$-algebra to the $(-q)$-degree part of its augmentation ideal. There is an evident ring homomorphism $\mathrm{End}(U^q) \to \mathrm{End}(I^q)$, and it is straightforward to show that this is an isomorphism. That is, $\Gamma^q$ is also the endomorphism ring of $I^q$.

3.14. **The rings $\Delta^q$.** The ring $\Delta^q$ is defined to be a ring which acts naturally on the degree $(-q)$-part of the *cotangent space* to an augmentation $A \to E_*$.

Let $Q^q\colon \mathrm{Alg}^*_{\mathbb{T}/E_*} \to \mathrm{Ab}$ denote the functor which sends an augmented $\mathbb{T}$-algebra $A \to E_*$ to the $(-q)$-th grading of its abelianization $\widetilde{A}/\widetilde{A}^2$, where $\widetilde{A}$ is the augmentation ideal. We *define* $\Delta^q \overset{\text{def}}{=} \mathrm{End}(Q^q)$, the endomorphism ring of the functor $Q^q$; thus an element $f \in \Delta^q$ gives a natural abelian group homomorphism $(\widetilde{A}/\widetilde{A}^2)_{-q} \to (\widetilde{A}/\widetilde{A}^2)_{-q}$ for all $A$ in $\mathrm{Alg}^*_{\mathbb{T}/E_*}$.

To each endomorphism $\phi\colon Q^q \to Q^q$ we associate an element $\phi(x)$ in the $E_*$-module $(\widetilde{\mathbb{T}}(E_*S^{-q})/(\widetilde{\mathbb{T}}(E_*S^{-q}))^2)_{-q}$, defined as the image of the canonical generator of $E_{-q}S^{-q}$ under the map

$$x \in E_{-q}S^{-q} \to \widetilde{\mathbb{T}}(E_*S^{-q})_{-q} \to Q^q(\mathbb{T}(E_*S^{-q})) \xrightarrow{\phi} Q^q(\mathbb{T}(E_*S^{-q}));$$

3.15. **Proposition.** *The map $\Delta^q \to Q^q(\mathbb{T}(E_*S^{-q})) \approx (\widetilde{\mathbb{T}}(E_*S^{-q}))/(\widetilde{\mathbb{T}}(E_*S^{-q}))^2)_{-q}$ sending $\phi$ to $\phi(x)$ is an isomorphism.*

*Proof.* It is straightforward to check, using naturality and the bijection $\widetilde{A}_{-q} = \mathrm{Hom}_{\mathrm{Alg}^*_{\mathbb{T}}/E_*}(\mathbb{T}(E_*S^{-q}), A)$, that an endomorphism $\phi$ is uniquely determined by the element $\phi(x)$. Thus it remains to show that the map of the proposition is surjective.

Next, suppose that $\phi$ is an endomorphism of the composite functor $\mathrm{Alg}^*_{\mathbb{T}/E_*} \xrightarrow{Q^q} \mathrm{Ab} \to \mathrm{Set}$. The abelian group structure $Q^q(A) \times Q^q(A) \to Q^q(A)$ is naturally isomorphic to the map obtained by applying $Q^q$ to the fold map $A \otimes_{E_*} A \to A$ of augmented $\mathbb{T}$-algebras. Thus, by naturality, $\phi$ must commute with this map. That is, every set endomorphism of $Q^q$ is automatically an abelian group endomorphism.

Given $y \in Q^q(\mathbb{T}(E_*S^{-q}))$, choose any lift $y \in \widetilde{\mathbb{T}}(E_*S^{-q})_{-q}$ and consider the corresponding endomorphism $\psi$ of the composite functor $\mathrm{Alg}^*_{\mathbb{T}}/E_* \xrightarrow{I^q} \mathrm{Ab} \to \mathrm{Set}$, which sends an augmented algebra to the underlying set of its augmentation ideal. Because $A \mapsto A/\widetilde{A}^2$ is a functor from $\mathrm{Alg}^*_{\mathbb{T}}/E_*$ to itself, we can apply $\psi$ to $I^q(A/\widetilde{A}^2) \approx Q^q(A)$ to obtain a natural abelian group endomorphism of $Q^q$, and a straightforward calculation shows that the evaluation of this endomorphism on the canonical generator is exactly $y$, as desired. □

Thus, the $E_0$-module $\Delta^q$ is isomorphic to the degree $-q$ part of the indecomposable quotient of the augmented $E_*$-algebra: $\mathbb{T}(E_*S^{-q})_{-q} \approx \bigoplus_{m\geq 0} E^\wedge_{-q}B\Sigma_m^{-q\rho_m}$. The rings $\Delta^q$

admit a grading $\Delta^q \approx \bigoplus_{k\geq 0} \Delta^q[k]$, where

$$\Delta^q[k] \approx \operatorname{Cok}\left[ \bigoplus_{0<j<p^k} E^\wedge_{-q} B(\Sigma_j \times \Sigma_{p^k-j})^{-q\rho_{p^k}} \to E^\wedge_{-q} B\Sigma_{p^k}^{-q\rho_{p^k}} \right],$$

the map being induced by the inclusion $\Sigma_j \times \Sigma_{p^k-j} \subset \Sigma_{p^k}$.

3.16. **Relation between $\Gamma^q$ and $\Delta^q$.** There are isomorphisms of functors $U^q \approx U^{q+2k}$ and $Q^q \approx Q^{q+2k}$ for all $k \in \mathbb{Z}$, because $E_*$ is an even periodic graded ring. The choice of such isomorphisms depends on a choice of isomorphism $E_* \approx E_{*+2k}$ of $E_*$-modules. We obtain the following consequence.

3.17. **Proposition.** *There are isomorphisms $\Gamma^q \approx \Gamma^{q+2k}$ and $\Delta^q \approx \Delta^{q+2k}$ of graded rings under $E_0$, for all $k \in \mathbb{Z}$.*

*Proof.* The only thing to note is that the isomorphisms $U^q \approx U^{q+2k}$ and $Q^q \approx Q^{q+2k}$ obtained from the periodicity of $E$ actually give isomorphisms of underlying $E_0$-modules, not merely of abelian groups. Thus the resulting ring isomorphisms are compatible with the inclusion of the subring $E_0$. □

Next, we will produce a chain of ring homomorphisms

$$\begin{array}{ccccc} \Gamma^{q+1} & & \Gamma^q & & \Gamma^{q-1} \\ {\scriptstyle f_{q+1}}\downarrow & \overset{g_{q+1}}{\nearrow} & \downarrow{\scriptstyle f_q} & \overset{g_q}{\nearrow} & \downarrow{\scriptstyle f_{q-1}} \\ \Delta^{q+1} & & \Delta^q & & \Delta^{q-1} \end{array}$$

Any natural operation $\phi\colon I^q \to I^q$ on augmentation ideals, applied to an augmented $\mathbb{T}$-algebra $A$, naturally induces an endomorphism $Q^q \to Q^q$ by passage to the indecomposables (i.e., evaluate $I^q$ on the quotient map $A \to A/\widetilde{A}^2$, which is a map of $\mathbb{T}$ algebras). Thus, we obtain a ring homomorphism $f_q\colon \Gamma^q \to \Delta^q$. Explicitly, the map $f_q[k]\colon \Gamma^q[k] \to \Delta^q[k]$ amounts to the natural map from the primitive subobject to the indecomposable quotient of $E^\wedge_{-q} B\Sigma_{p^k}^{-q\rho_{p^k}}$.

3.18. **Proposition.** *The map $f_q\colon \Gamma^q \to \Delta^q$ is an isomorphism if $q$ is odd.*

*Proof.* The argument of the proof of [Rez09, Prop. 7.2] shows that $\mathbb{T}(E_* S^{-q}) \approx \bigoplus \pi_* L_{K(n)} \mathbb{P}_m \Sigma^{-q} E$ is, as a Hopf algebra, a primitively generated exterior algebra when $q$ is odd. Thus, the evident map from primitives to indecomposables is an isomorphism. □

Recall that the loop construction (§3.11) gives a functor $\Omega\colon \mathrm{Alg}^*_{\mathbb{T}/E_*} \to (\mathrm{Alg}^*_{\mathbb{T}/E_*})_{\mathrm{ab}} \subset \mathrm{Alg}^*_{\mathbb{T}/E_*}$. Thus, there is a natural isomorphism of abelian groups $Q^q(\Omega A) \approx I^{q-1}(A)$, and hence any endomorphism of $Q^q$ induces an endomorphism of $I^{q-1}$. We have thus defined a ring homomorphism $g_q\colon \Delta^q \to \Gamma^{q-1}$. Explicitly, the map $g_q[k]\colon \Delta^q[k] \to \Gamma^{q-1}[k]$ is induced by the "suspension" map

$$E^\wedge_{-q} B\Sigma_{p^k}^{-q\rho_{p^k}} \to E^\wedge_{-q+1} B\Sigma_{p^k}^{(-q+1)\rho_{p^k}},$$

which factors through the quotient $\Delta^q[k]$ and lands in the submodule $\Gamma^{q-1}[k]$.

3.19. **Proposition.** *The map $g_q\colon \Delta^q \to \Gamma^{q-1}$ is an isomorphism for all $q$.*

*Proof.* In light of (3.13) and (3.18), it is enough to show that the composite

$$\Delta^{2q}[k] \xrightarrow{g_{2q}} \Gamma^{2q-1}[k] \xrightarrow[\sim]{f_{2q-1}} \Delta^{2q-1}[k] \xrightarrow{g_{2q-1}} \Gamma^{2q-2}[k]$$

is an isomorphism. Because $E_*$ is 2-periodic, it is enough to consider the case $q = 0$. Explicitly, we need to show that if we apply completed $E_*$-homology to the "zero-section" map

$$B\Sigma_{p^k}^+ \to B\Sigma_{p^k}^{2\bar{\rho}_{p^k}} \approx \Sigma^{-2} B\Sigma_{p^k}^{2\rho_{p^k}},$$

the image is exactly the submodule $\Gamma^{-2}[k]$. (Here $\bar{\rho}_{p^k}$ denotes the reduced real representation, so that $\rho_{p^k} = \mathbb{R} \oplus \bar{\rho}_{p^k}$.) That this is the case follows by Theorems 8.5 and 8.6 of [Str98], where the result is stated in "dual" form. Specifically, Strickland proves that $\mathrm{Prim} E^0 B\Sigma_{p^k} \to \mathrm{Ind} E^0 B\Sigma_{p^k}$ is generated by the Euler class of $\bar{\rho}_{p^k} \otimes \mathbb{C} \approx 2\bar{\rho}_{p^k}$, where Prim is the kernel of restrictions, and Ind the quotient of transfers, along $\Sigma_i \times \Sigma_{p^k-i} \subset \Sigma_{p^k}$. □

3.20. **Corollary.** *All of the rings $\Gamma^q$ and $\Delta^q$ are isomorphic as graded rings under $E_0$. Furthermore, each of the modules $\Gamma^q[k]$ and $\Delta^q[k]$ are finitely generated free $E_0$-modules.*

*Proof.* By the above, we have isomorphisms $\Delta^0 \xrightarrow{g_0} \Gamma^{-1} \xrightarrow{f_{-1}} \Delta^{-1} \xrightarrow{g_{-1}} \Gamma^{-2}$, and the general isomorphism follows by (3.17). The freeness follows from (3.13). □

Finally, we have the following result about the rank of the free $E_0$-modules $\Gamma[k]$ (and hence of $\Delta[k]$).

3.21. **Proposition.** *The ranks of $\Gamma[k]$ and $\Delta[k]$ (as left $E_0$-modules) are given by the generating series*

$$\sum_k \operatorname{rank} \Gamma[k] \cdot T^k = \left[(1-T)(1-pT)\cdots(1-p^{n-1}T)\right]^{-1},$$

*where $n$ is the height of the formal group associated to $E$.*

*Proof.* As noted in the proof of (3.13), $\Gamma[k] \subseteq E_0^\wedge B\Sigma_{p^k}$ is the inclusion of a summand of a free module. Taking $E_0$-module duals gives a projection $E^0 B\Sigma_{p^k} \to E^0 B\Sigma_{p^k}/I_{p^k}$ which admits an $E_0$-module retraction. Here $I_{p^k}$ is the ideal generated by transfers from subgroups $\Sigma_r \times \Sigma_{p^k-r}$ for $0 < r < p^k$.

The $E_0$-rank of $\Gamma[k]$ is equal to the $E_0$-rank of $E^0 B\Sigma_{p^k}/I_{p^k}$. Strickland [Str98, Thm. 1.1] computes its rank over $E_0$ as

$$\operatorname{rank} E^0 B\Sigma_{p^k}/I_{p^k} = |\{\text{subgroups } A \leq (\mathbb{Q}_p/\mathbb{Z}_p)^n \text{ with } |A| = p^k\}|.$$

A standard combinatorial argument gives the generating series.

□

For the remainder of this paper, we write $\Delta$ for the ring $\Delta^0$; this is the ring we will show is Koszul.

3.22. **Aside: $(\Gamma^q)^\wedge_{\mathfrak{m}}$ is the ring of additive operations on the $\pi_*$ of $K(n)$-local $E$-algebras.** We here describe the relation between the rings $\Gamma^q$ defined above, and the rings of additive operations on homotopy groups of $K(n)$-local commutative $E$-algebras.

An **additive operation** in degree $-q$ is a natural endomorphism of the functor

$$\pi_{-q}\colon h\mathrm{Alg}_{E,K(n)} \to \mathrm{Ab}$$

which computes the $(-q)$th homotopy group of a $K(n)$-local commutative $E$-algebra. The ring of additive operations in degree $-q$ is $\mathrm{End}(\pi_{-q})$ . The functor $\pi_{-q}$ (as a functor to sets) is corepresented by the object $L_{K(n)}\mathbb{P}(\Sigma^{-1}E)$, whence

$$\mathrm{End}(\pi_{-q}) \approx \mathrm{Ker}\bigl[\pi_{-q}L_{K(n)}\mathbb{P}(\Sigma^{-q}E) \to \pi_{-q}L_{K(n)}\mathbb{P}(\Sigma^{-q}E \vee \Sigma^{-q}E)\bigr],$$

where the map is induced by the diagonal map $\Sigma^{-q}E \to \Sigma^{-q}E \vee \Sigma^{-q}E$.

3.23. **Proposition.** *As rings under $E_0$, $\mathrm{End}(\pi_{-q}) \approx (\Gamma^q)^\wedge_{\mathfrak{m}}$.*

*Proof.* Recall that by definition $\Gamma^q$ sits in an exact sequence

$$0 \to \Gamma^q \xrightarrow{i} \mathbb{T}(E_*S^{-q})_{-q} \xrightarrow{j} \mathbb{T}(E_*S^{-q} \oplus E_*S^{-q})_{-q}.$$

We will show that this sequence remains exact after taking $\mathfrak{m}$-adic completion. By the isomorphism of (3.6)(4) and the exact sequence defining $\mathrm{End}(\pi_{-q})$ given above, the proposition follows.

To prove the claim, note that the map $i$ is split (as observed in the proof of (3.13), whence it is an inclusion of a direct sum decomposition $\mathbb{T}(E_*S^{-1})_{-q} \approx \Gamma^q \oplus N$, and $j$ factors through an inclusion $N \to \mathbb{T}(E_*S^{-q} \oplus E_*S^{-q})_{-q}$. Recall that $\mathbb{T}$ applied to a free $E_*$-module is a free module, whence all terms in the sequence, as well as $N$, are free $E_0$-modules.

Thus, to show that applying $\mathfrak{m}$-adic completion to the above sequence yields another exact sequence, it suffices to show that the map $N^\wedge_{\mathfrak{m}} \to (\mathbb{T}(E_*S^{-q} \oplus E_*S^{-q})_{-q})^\wedge_{\mathfrak{m}}$ induced by $j$ is injective. In fact, if $j\colon N \to N'$ is any monomorphism of *free* $E_0$-modules, then the induced map on $\mathfrak{m}$-adic completions also a monomorphism. To see this, simply note that the completion of $\bigoplus_{i\in I} E_0$ can be explictly identified with the set of tuples $(a_i) \in \prod_{i\in I} E_0$ such that for each $k \geq 0$, $a_i \in \mathfrak{m}^k$ for all but finitely many $i$.

□

## 4. Koszul rings

In this section, we develop the theory of Koszul rings in terms of the bar construction (following the original [Pri70]), and in the generality we need. Specifically, we describe the theory for a ring $A$ which contains a commutative ring $R$, but which is *not central* $A$. I believe these results are standard, but I do not know a convenient reference in the literature. In any case, we need to set up the interpretation of the Koszul property in terms of the bar construction, for our results in §7.

Furthermore, we will show that a ring $A$ which is Koszul in our sense is necessarily *quadratic* (4.10). Once we show that the ring $\Delta$ is Koszul in our sense, we will have thus proved that it is quadratic.

In the following let $A = \bigoplus_{k\geq 0} A[k]$ be a graded associative ring, and suppose that $R = A[0]$ is commutative. It is important that we do *not* assume that $R$ is central in $A$. We write $\epsilon\colon A \to R$ for the evident augmentation map.

4.1. **Bar constructions for rings.** Let $M$ be a right $A$-module, and let $N$ be a left $A$-module. The **two-sided bar construction** $\mathcal{B}(M, A, N)$ is the simplicial abelian group defined by

$$\mathcal{B}_q(M, A, N) = M \otimes_R \underbrace{A \otimes_R \cdots \otimes_R A}_{q \text{ copies}} \otimes_R N,$$

with face and boundary maps defined in the usual way.

Let $\mathcal{N}\mathcal{B}(M, A, N)$ denote the *normalized* chain complex obtained from the bar resolution, (obtained by quotienting out by the image of degeneracy operators); we have $H_*\mathcal{N}\mathcal{B}(M, A, N) = H_*\mathcal{B}(M, A, N)$.

Let $\overline{\mathcal{B}}(A) = \mathcal{B}(R, A, R)$, where $R$ is viewed as a left or right $A$-module using the projection $\epsilon\colon A \to A[0] \approx R$ defined by $\epsilon(A[m]) = 0$ for $m > 0$. The complex $\overline{\mathcal{B}}(A)$ inherits a grading from the grading on $A$, so that there is an isomorphism of complexes $\overline{\mathcal{B}}(A) \approx \bigoplus \overline{\mathcal{B}}(A)[m]$, where

$$\overline{\mathcal{B}}_q(A)[m] \approx \bigoplus_{m_1+\cdots+m_q=m} A[m_1] \otimes_R \cdots \otimes_R A[m_q],$$

where each index $m_i > 0$. Thus the homology of $\overline{\mathcal{B}}(A)$ is graded, too:

$$H_*\overline{\mathcal{B}}(A) \approx \bigoplus_{m\geq 0} H_*\overline{\mathcal{B}}(A)[m].$$

4.2. **Proposition.** *We have that*

$$H_0\overline{\mathcal{B}}(A) \approx A[0] = R,$$

*and that*

$$H_q\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for } q > m.$$

*Proof.* The normalized complex $\mathcal{N}\overline{\mathcal{B}}(A)$ is such that $\mathcal{N}\overline{\mathcal{B}}(A)_q[m] = 0$ if $q > m$, or if $q = 0$ and $m > 0$. □

Let $\mathcal{B}(A) \stackrel{\text{def}}{=} \mathcal{B}(A, A, A)$, the "big" two-sided bar construction on $A$. Since $\epsilon\colon A \to R$ is a map of $A$-bimodules, there is an induced surjective map $\mathcal{B}(A) \to \overline{\mathcal{B}}(A)$ of complexes. Let $\widetilde{A} = \ker \epsilon$; this is an $A$-bimodule, so we can define complexes

$$\hat{\mathcal{B}}(A) \stackrel{\text{def}}{=} \mathcal{B}(A, A, \widetilde{A}), \quad \check{\mathcal{B}}(A) \stackrel{\text{def}}{=} \mathcal{B}(\widetilde{A}, A, A), \quad \ddot{\mathcal{B}}(A) \stackrel{\text{def}}{=} \mathcal{B}(\widetilde{A}, A, \widetilde{A}),$$

each of which is naturally a subcomplex of $\mathcal{B}(A)$. (The notation is meant to align with the analogous notation for certain subcomplexes of the nerve of the partition poset, see (6.2).)

4.3. **Proposition.** *The sequence of complexes*

$$0 \to \ddot{\mathcal{B}}(A) \xrightarrow{(incl., -incl.)} \hat{\mathcal{B}}(A) \oplus \check{\mathcal{B}}(A) \xrightarrow{(incl., incl.)} \mathcal{B}(A) \to \overline{\mathcal{B}}(A) \to 0$$

*is exact.*

*Proof.* In degree $q \geq 0$, $\mathcal{B}(A)_q$ has the form

$$\mathcal{B}(A)_q \approx \bigoplus_{m_0,\ldots,m_{q+1}\geq 0} A[m_0] \otimes_R \cdots \otimes_R A[m_{q+1}].$$

It is straightforward to identify the subgroups $\hat{\mathcal{B}}(A)_q$, $\check{\mathcal{B}}(A)_q$, and $\ddot{\mathcal{B}}(A)_q$ as consisting of those summands with either $m_{q+1} > 0$, $m_0 > 0$, or both, as the case may be. The result follows easily. □

4.4. **Definition of Koszul rings.** We say that the graded ring $A$ is **Koszul** if

$$H_q\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for } q < m.$$

In view of the previous proposition, this means that if $A$ is Koszul, then $\overline{\mathcal{B}}(A)[m]$ is a chain complex of $R$-modules whose homology is concentrated in the single dimension $m$.

4.5. *Example.* Let $V$ be an $R$-bimodule. The **tensor algebra** $TV$ is defined by

$$TV \overset{\text{def}}{=} \left( \bigoplus_{m\geq 0} \underbrace{V \otimes_R \cdots \otimes_R V}_{m \text{ copies}} \right).$$

It is a straightforward exercise to show that $H_q\overline{\mathcal{B}}(TV)[m] = 0$ for all $q \geq 0$ and all $m > 1$. Thus $TV$ is Koszul.

For a Koszul ring $A$, let

$$C[m] \overset{\text{def}}{=} H_m\overline{\mathcal{B}}(A)[m].$$

4.6. **Proposition.** *Suppose that $A$ is Koszul, and that each $A[m]$ is finitely generated and projective as a left $R$-module. Then each $C[m]$ is finitely generated and projective as a left $R$-module.*

*Suppose each $A[m]$ is as above, and we write* $\operatorname{rank}_{\mathfrak{P}} A[m] = a_m$ *for the rank of $A[m]_{\mathfrak{P}}$ as an $R_{\mathfrak{P}}$-module for some prime $\mathfrak{P} \subset R$. Then each $C[m]$ has*

$$\sum_{m=0}^{\infty} \operatorname{rank}_{\mathfrak{P}} C[m] \cdot T^m = \left( \sum_{m=0}^{\infty} (-1)^m a_m \cdot T^m \right)^{-1}$$

*in* $\mathbb{Z}[\![T]\!]$.

*Proof.* This is straightforward, using the fact that if $P$ and $Q$ are finitely generated and projective as left $R$-modules, then so is $P \otimes_R Q$, and that if $R$ is a local ring, $\operatorname{rank} P \otimes_R Q = (\operatorname{rank} P)(\operatorname{rank} Q)$. □

4.7. **Koszul resolutions.** Let $M$ be a right, and $N$ a left $R$-module. Define a filtration $\{F_m\}$ of the normalized bar complex $\mathcal{N}\mathcal{B}(M, A, N)$, so that

$$F_m = F_m\mathcal{N}\mathcal{B}_q(M, A, N) \overset{\text{def}}{=} M \otimes_R \left( \bigoplus_{m_1+\cdots+m_q \leq m} A[m_1] \otimes_R \cdots \otimes_R A[m_q] \right) \otimes_R N.$$

There are isomorphisms of complexes

$$F_m/F_{m-1} \approx M \otimes_R \overline{\mathcal{B}}(A)[m] \otimes_R N,$$

and thus a spectral sequence $E_1^{p,q} = H_p(M \otimes_R \overline{\mathcal{B}}(A)[q] \otimes_R N) \Rightarrow H_{p+q}(\mathcal{B}(M, A, N))$. This immediately gives the following.

4.8. **Proposition.** *Suppose that $A$ is Koszul. If $M$ and $N$ are flat as right and left $R$-modules respectively, then the $E_1$-term of the above spectral sequence collapses to a chain complex of the form*

$$\cdots \to M \otimes_R C[m] \otimes_R N \to M \otimes_R C[m-1] \otimes_R N \to \cdots .$$

This is the **Koszul complex** for computing $\mathrm{Tor}^A(M,N)$. Taking $M = A$ provides for any flat (resp. projective) $N$ a flat (resp. projective) $A$-module **Koszul resolution**

$$\cdots \to A \otimes_R C[m] \otimes_R N \to A \otimes_R C[m-1] \otimes_R N \to \cdots \to A \otimes_R C[0] \otimes_R N \to N \to 0,$$

which is an exact sequence of left $A$-modules.

4.9. **Koszul rings are quadratic.** A graded ring $A$ is *quadratic* if it is generated as a ring under $R$ by elements of degree one, and if all relations are generated by the homogeneous relations of degree 2. That is, $A$ is quadratic if the natural map

$$T(A[1])/I \to A$$

is an isomorphism, where $T(V)$ represents the associative ring over $R$ freely generated by an $R$-bimodule $V$, and $I$ is a two-sided ideal generated by a sub-bimodule $P \subseteq A[1] \otimes_R A[1] \subset T(A[1])$.

4.10. **Proposition.** *If $A$ is Koszul, and is flat as a left $R$-module, then $A$ is quadratic.*

This result is well-known in the case that $R$ is central; see, for instance, [PP05]. This will follow from the sharper result (4.12) below.

4.11. **Proposition.** *The ring $A$ is generated by $A[1]$ if and only if $H_1\overline{\mathcal{B}}(A)[m] = 0$ for all $m > 1$.*

*Proof.* Since $\overline{\mathcal{B}}_0(A)[m] = 0$ for $m > 0$, we have for each $m \geq 2$ an exact sequence

$$\bigoplus_{\substack{k+\ell=m \\ k,\ell>0}} A[k] \otimes_R A[\ell] \xrightarrow{\text{mult}} A[m] \to H_1\overline{\mathcal{B}}(A)[m] \to 0.$$

Clearly, $H_1\overline{\mathcal{B}}(A)[m] = 0$ if and only if $A[m]$ is spanned by products of pairs of elements of strictly lower degree. The result follows. □

4.12. **Proposition.** *Suppose $A$ is flat as a left $R$-module. Then $A$ is quadratic if and only if*

$$H_1\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for all } m > 1,$$

*and*

$$H_2\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for all } m > 2.$$

*Proof.* By the previous proposition, it suffices to show that if $A$ is generated over $R$ by $A[1]$, then $A$ is quadratic if and only if $H_2\overline{\mathcal{B}}(A)[m] = 0$ for all $m > 2$.

Let $f\colon T(A[1]) \to A$ denote the homomorphism of rings which is the identity map on $A[1]$; it is surjective and grading preserving. Consider the resulting exact sequence of chain complexes

$$0 \to K[m] \xrightarrow{\gamma} \overline{\mathcal{B}}(TA[1])[m] \xrightarrow{g} \overline{\mathcal{B}}(A)[m] \to 0,$$

where $g$ is induced by the algebra map $f$. Examining the exact sequence of complexes in degrees 1 and 2 gives the commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
\bigoplus_{\substack{p+q=m \\ p,q>0}} \big(K_1[p]\otimes_R A[1]^{\otimes q}\big) \oplus \big(A[1]^{\otimes p}\otimes_R K_1[q]\big) & \xrightarrow{\beta_m} & K_2[m] & \xrightarrow{\delta_m} & K_1[m] \\
 & \searrow^{\widetilde{\beta}=\bigoplus(\gamma\otimes\mathrm{id},\,\mathrm{id}\otimes\gamma)} & \downarrow & & \downarrow{\scriptstyle \gamma} \\
 & & \bigoplus_{\substack{p+q=m \\ p,q>0}} A[1]^{\otimes p}\otimes_R A[1]^{\otimes q} & \longrightarrow & A[1]^{\otimes m} \\
 & & \downarrow{\scriptstyle g_2=\bigoplus f\otimes f} & & \downarrow{\scriptstyle f} \\
 & & \bigoplus_{\substack{p+q=m \\ p,q>0}} A[p]\otimes_R A[q] & \longrightarrow & A[m] \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

in which the columns are exact. Since $g_2 \circ \widetilde{\beta} = 0$, there is a unique lift $\beta_m$ as shown in the diagram, and $\beta_m$ must be surjective by the flatness hypothesis on $A$.

We have already observed that the complex $\overline{\mathcal{B}}(TA[1])[m]$ is acyclic for all $m > 1$ (4.5). Thus $H_2\overline{\mathcal{B}}(A)[m] \approx H_1K[m]$ for $m > 1$, and since $K_0[m] = 0$, we obtain an exact sequence

$$K_2[m] \xrightarrow{\delta_m} K_1[m] \to H_2\overline{\mathcal{B}}(A)[m] \to 0.$$

Putting this all together, we find that (for $m > 2$), $H_2\overline{\mathcal{B}}(A)[m] = 0$ if and only if $\delta_m$ is surjective, if and only if $\delta_m\beta_m$ is surjective. But $\delta_m\beta_m$ being surjective means exactly that $K_1[m]$ (the module of relations of $A$ of degree $m$) is generated by relations of lower degree. □

## 5. Linearization of functors

In this section, we discuss a certain "linearization" operation on functors between abelian categories. The linearization construction gives us a formal approach to the algebra $\Delta$ of §3, and will be used later to obtain a bar-resolution of $\Delta$ from partition complexes.

5.1. **The linearization construction.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{B}$ be an abelian category. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a (not necessarily additive) functor. We say $F$ is **reduced** if $F0 \approx 0$.

For such a reduced functor $F$, we define $\mathcal{L}_F\colon \mathcal{A} \to \mathcal{B}$ together with a natural transformation $\epsilon\colon F \to \mathcal{L}_F$ by setting $\mathcal{L}_F(X)$ to be the coequalizer of

$$F(X \oplus X) \underset{F(p_1)+F(p_2)}{\overset{F(p_1+p_2)}{\rightrightarrows}} F(X)$$

where $p_i\colon X\oplus X\to X$ for $i=1,2$ are the two projections.

5.2. **Proposition.** *The functor $\mathcal{L}_F$ is additive; any natural map $F\to G$ to an additive functor $G$ factors uniquely through $\epsilon\colon F\to\mathcal{L}_F$.*

*Proof.* For the first part, note that if $f,g\colon Y\to X$ are two maps in $\mathcal{A}$, then

$$\epsilon F(f+g)=\epsilon F(p_1+p_2)F((f,g))=\epsilon(F(p_1)+F(p_2))F((f,g))=\epsilon(F(f)+F(g)),$$

as maps $F(Y)\to\mathcal{L}_F(X)$, from which it follows that $\mathcal{L}_F(f+g)=\mathcal{L}_F(f)+\mathcal{L}_F(g)$. The second part follows from the observation that $\mathcal{L}_G\approx G$ when $G$ is additive. ☐

Let $\perp F(X)\overset{\text{def}}{=}\ker\left[F(X\oplus X)\xrightarrow{(F(p_1),F(p_2))}F(X)\oplus F(X)\right]$, and write $\beta_F\colon\perp F(X)\to F(X\oplus X)$ for the inclusion. Then there is an exact sequence

$$\perp F(X)\xrightarrow{\gamma_F}F(X)\xrightarrow{\epsilon}\mathcal{L}_F(X)\to 0.$$

where $\gamma_F=F(p_1+p_2)\circ\beta_F$.

5.3. **A "chain rule".** Given functors $F\colon\mathcal{A}\to\mathcal{B}$ and $G\colon\mathcal{B}\to\mathcal{C}$, where $\mathcal{A}$ is additive and $\mathcal{B}$ and $\mathcal{C}$ are abelian, there is a unique natural transformation $c\colon\mathcal{L}_{F\circ G}\to\mathcal{L}_F\circ\mathcal{L}_G$ such that $c\epsilon_{F\circ G}=\epsilon_F\circ\epsilon_G\colon F\circ G\to\mathcal{L}_F\circ\mathcal{L}_G$, since $\mathcal{L}_F\circ\mathcal{L}_G$ is additive. Our chain rule says that the transformation $c$ is an isomorphism whenever things split.

5.4. **Proposition.** *Let $X$ be an object of $\mathcal{A}$. Suppose that there are direct sum decompositions*

$$A\oplus B\xrightarrow[\sim]{(i_A,i_B)}G(X),\qquad B\oplus C\xrightarrow[\sim]{(i'_B,i_C)}\perp G(X),$$

*such that $\epsilon_G i_A\colon A\to\mathcal{L}_G(X)$ is an isomorphism, $\gamma_G i'_B=i_B$, and $\gamma_G i_C=0=\epsilon_G i_B$. Then the natural map $c\colon\mathcal{L}_{F\circ G}(X)\to\mathcal{L}_F(\mathcal{L}_G(X))$ is an isomorphism.*

5.5. **Corollary.** *Let $F,G\colon\mathcal{A}\to\mathcal{A}$ be functors on an abelian category $\mathcal{A}$, and suppose that $G(X)$ and $\mathcal{L}_G(X)$ are projective whenever $X$ is a projective object. Then $c\colon\mathcal{L}_{F\circ G}(X)\to\mathcal{L}_F(\mathcal{L}_G(X))$ is an isomorphism for all projective $X$.*

The proof of (5.4) is given at the end of this section.

5.6. **The ring $\Delta$ is the linearization of the monad $\widetilde{\mathbb{T}}$.**

5.7. **Proposition.** *Let $m=p^k$, and let $M$ be a free $E_*$-module concentrated in even degree. Then the natural map $\mathcal{L}_{\mathbb{T}\langle m\rangle}(M)\to\Delta[k]\otimes_{E_0}M$ is an isomorphism. If $m\neq p^k$, $\mathcal{L}_{\mathbb{T}\langle m\rangle}(M)=0$. These maps fit together to give an isomorphism $\mathcal{L}_{\widetilde{\mathbb{T}}}(M)\approx\Delta\otimes_{E_0}M$.*

*Proof.* Since $\mathbb{T}$ commutes with filtered colimits (§3.5(6)), so does $\mathcal{L}_{\mathbb{T}\langle m\rangle}$, and so it is enough to consider finite free modules $M=(E_*)^k$. Since both $\mathcal{L}_{\mathbb{T}\langle m\rangle}$ and $\Delta\otimes_{E_*}(-)$ are additive, it is enough to consider the module $M=E_*$. Now we compute that for $m\geq 1$,

$$\begin{aligned}\perp\mathbb{T}\langle m\rangle(E_*)&=\operatorname{Ker}\left[\mathbb{T}\langle m\rangle(E_*\oplus E_*)\xrightarrow{(\mathbb{T}\langle m\rangle p_1,\mathbb{T}\langle m\rangle p_2)}\mathbb{T}\langle m\rangle(E_*)\oplus\mathbb{T}\langle m\rangle(E_*)\right]\\&\approx\bigoplus_{0<i<m}E_*B(\Sigma_i\times\Sigma_{m-i}).\end{aligned}$$

The map $\mathbb{T}\langle m\rangle(p_1+p_2)\colon \mathbb{T}\langle m\rangle(E_*\oplus E_*)\to \mathbb{T}\langle m\rangle(E_*)$ computes the effect of $\mathbb{P}_m(p_1+p_2)\colon \mathbb{P}_m(E\vee E)\to \mathbb{P}_m(E)$ on homotopy, and we see that $\mathcal{L}_{\mathbb{T}\langle m\rangle}(E_*)$ is the cokernel of $\bigoplus_{0<i<m} E_*B(\Sigma_i\times\Sigma_{m-i})\to E_*B\Sigma_m$ induced by transfers. This map is surjective if $m$ is not a power of $p$; if $m=p^k$ then in degree 0 the cokernel is precisely $\Delta[k]$, and in general degree the cokernel is $\Delta[k]\otimes_{E_0}E_*$. □

In particular, it follows that $\mathcal{L}_{\widetilde{\mathbb{T}}}(M)$ is a free $E_*$-module whenever $M$ is, since $\Delta$ is a free $E_0$-module (3.20).

5.8. **Proposition.** *There is an isomorphism*

$$\mathcal{L}_{\mathcal{B}(\widetilde{\mathbb{T}})}(M)\approx \mathcal{B}(\Delta)\otimes_{E_0} M$$

*of simplicial $E_*$-modules, which is compatible with grading, so that $\mathcal{L}_{\mathcal{B}(\widetilde{\mathbb{T}})\langle m\rangle}(M)\approx \mathcal{B}(\Delta)[k]\otimes_{E_0} M$, where $m=p^k$.*

*Proof.* If $M$ is a free module, then $\widetilde{\mathbb{T}}(M)$ and $\mathcal{L}_{\widetilde{\mathbb{T}}(M)}$ are free. Thus, the chain rule (5.5) applies to show that $\mathcal{L}_{\widetilde{\mathbb{T}}^{\circ q}}(M)\approx \mathcal{L}^{\circ q}_{\widetilde{\mathbb{T}}}(M)$ is an isomorphism for $q\geq 0$. The result follows using (5.7). □

We can make a more refined statement about this isomorphism: all contributions to $\mathcal{L}_{\mathcal{B}(\widetilde{\mathbb{T}})}$ come from the pure weight part (§2.2) of $\widetilde{\mathbb{T}}^{\circ q}\langle m\rangle$.

5.9. **Proposition.** *For all $m=m_1\cdots m_q$ with $m_i=p^{k_i}$ and $k_i\geq 0$, the diagram*

$$\begin{array}{ccc}
\mathcal{L}_{\mathbb{T}\langle m_1\rangle\circ\cdots\mathbb{T}\langle m_q\rangle}M & \longrightarrow & \Delta[k_1]\otimes_{E_0}\cdots\otimes_{E_0}\Delta[k_q]\otimes_{E_0}M \\
\downarrow & & \downarrow \\
\mathcal{L}_{\widetilde{\mathbb{T}}^{\circ q}}(M) & \longrightarrow & \Delta\otimes_{E_0}\cdots\otimes_{E_0}\Delta\otimes_{E_0}M
\end{array}$$

*commutes.*

*Proof.* This is straightforward from the naturality of the linearization construction. □

5.10. **Proof of the chain rule.** We give here a tedious elementary proof of (5.4); afterwards, we indicate how a somewhat more conceptual proof may be constructed, using the results of [JM04].

*Proof.* First we claim that $c\colon \mathcal{L}_{F\circ G}(X)\to\mathcal{L}_F(\mathcal{L}_G(X))$ is an epimorphism. The commutative diagram

$$\begin{array}{ccccc}
F(A) & \xrightarrow{F(i_A)} & F(G(X)) & \xrightarrow{\epsilon_{F\circ G}} & \mathcal{L}_{F\circ G}(X) \\
 & \underset{F(\epsilon_G i_A)}{\overset{\sim}{\searrow}} & \downarrow{\scriptstyle F(\epsilon_G)} & & \downarrow{\scriptstyle c} \\
 & & F(\mathcal{L}_G(X)) & \xrightarrow[\epsilon_F]{} & \mathcal{L}_F(\mathcal{L}_G(X))
\end{array}$$

shows that $c$ factors through an epimorphism $F(A)\to\mathcal{L}_F(\mathcal{L}_G(X))$, and thus is an epimorphism.

Let $D = \operatorname{Ker}\left[F(A \oplus B) \xrightarrow{(F(p_1),F(p_2))} F(A) \oplus F(B)\right]$, and write $\alpha\colon D \to F(A \oplus B)$ for the inclusion. Let $i_D\colon D \to F(G(X))$ be defined by $F((i_A, i_B)) \circ \alpha$. The commutative diagram

$$\begin{array}{ccccc}
F(A) \oplus F(B) \oplus D & \xrightarrow[\sim]{(Fi_A, Fi_B, i_D)} & F(G(X)) & \xrightarrow{\epsilon_{F\circ G}} & \mathcal{L}_{F\circ G}(X) \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle F\epsilon_G}\downarrow & & \downarrow{\scriptstyle c} \\
F(A) & \xrightarrow[F(\epsilon_G i_A)]{\sim} & F(\mathcal{L}_G(X)) & \xrightarrow[\epsilon_F]{} & \mathcal{L}_F(\mathcal{L}_G(X))
\end{array}$$

shows that we can identify the kernel of the projection $F(G(X)) \to \mathcal{L}_F(\mathcal{L}_G(X))$ with the image of the bottom horizontal map in

$$\begin{array}{ccc}
 & & F(G(X \oplus X)) \\
 & \overset{(g_1,g_2,g_3)}{\nearrow} & \downarrow{\scriptstyle F(G(p_1+p_2)) - F(G(p_1)) - F(G(p_2))} \\
\perp F(A) \oplus F(B) \oplus D & \xrightarrow[((Fi_A)\gamma_F, Fi_B, i_D)]{} & F(G(X))
\end{array}$$

We will construct a section $(g_1, g_2, g_3)$ making the above diagram commute, and thus prove that the projection $F(G(X)) \to \mathcal{L}_F(\mathcal{L}_G(X))$ factors through an isomorphism $c\colon \mathcal{L}_{F\circ G}(X) \to \mathcal{L}_F(\mathcal{L}_G(X))$, as desired.

Let $g_1\colon \perp F(A) \to F(G(X \oplus X))$ be defined by the composite

$$\perp F(A) \xrightarrow{\beta_F} F(A \oplus A) \xrightarrow{F(i_A \oplus i_A)} F(G(X) \oplus G(X)) \xrightarrow{F((G(i_1),G(i_2)))} F(G(X \oplus X)).$$

It is straightforward to check that

$$F(G(p_\alpha)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_A) = F(i_A) \circ F(p_\alpha), \qquad (\alpha = 1, 2)$$
$$F(G(p_1 + p_2)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_A) = F(i_A) \circ F(p_1 + p_2).$$

Thus

$$(F(G(p_1 + p_2)) - F(G(p_1)) - F(G(p_2))) \circ g_1 = F(i_A) \circ \gamma_F$$

as desired.

Let $g_2\colon F(B) \to F(G(X \oplus X))$ be defined by the composite

$$F(B) \xrightarrow{F(i'_B)} F(\perp G(X)) \xrightarrow{F\beta_G} F(G(X \oplus X)).$$

Since

$$F(G(p_\alpha)) \circ F(\beta_G) = 0, \quad (\alpha = 1, 2), \qquad F(G(p_1 + p_2)) \circ F(\beta_G) = F(\gamma_G),$$

we see that

$$(F(G(p_1 + p_2) - F(G(p_1)) - F(G(p_2))) \circ g_2 = F(\gamma_G) \circ F(i'_B) = F(i_B),$$

as desired.

Let $g_3\colon D \to F(G(X \oplus X))$ be defined by the composite

$$D \xrightarrow{\alpha} F(A \oplus B) \xrightarrow{F(i_A \oplus i_B)} F(G(X) \oplus G(X)) \xrightarrow{F((G(i_1),G(i_2)))} F(G(X \oplus X)).$$

It is straightforward to check that

$$F(G(p_1)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_B) = F(i_A) \circ F(p_1),$$
$$F(G(p_2)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_B) = F(i_B) \circ F(p_2),$$
$$F(G(p_1 + p_2)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_B) = F((i_A, i_B)).$$

Thus

$$(F(G(p_1 + p_2)) - F(G(p_1)) - F(G(p_2))) \circ g_3 = F(i_D),$$

as desired. □

5.11. **Proof of the chain rule, using Johnson-McCarthy.** We briefly describe how one may produce a proof of the chain-rule using the work of [JM04]. In that paper, the authors describe a "derived linearization" construction which, given a functor $F\colon \mathcal{A} \to \mathcal{B}$ from an additive category to an abelian category such that $F0 = 0$, produces a functor $D_1F\colon \mathcal{A} \to \mathrm{Ch}\mathcal{B}$ to the category of chain complexes in $\mathcal{B}$. In degrees 1 and 0, the chain complex $D_1F$ has the form

$$\cdots \to \perp F \xrightarrow{\gamma_F} F,$$

and thus $\mathcal{L}_F = H_0 D_1 F$.

According to [JM04, Lemma 5.7], there is a quasi-isomorphism $D_1F \circ D_1G \approx D_1(F \circ G)$, where the left-hand side is the total complex of the bicomplex obtained by applying $D_1F$ degreewise to $D_1G$. Thus, to recover (5.4) from their result, it is necessary to show that $H_0(D_1F \circ D_1G) \approx H_0D_1F \circ H_0D_1G$. This is where the hypotheses on $G(X)$ come into play. The functor $D_1F$ is additive up to quasi-isomorphism (i.e., $D_1F(X) \oplus D_1F(Y) \to D_1F(X \oplus Y)$ is always a quasi-isomorphism). Thus, under the hypotheses of (5.4), applying $D_1F$ to the sequence $\perp G(X) \to G(X)$ gives a map quasi-isomorphic to

$$D_1F(B) \oplus D_1F(C) \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} D_1F(A) \oplus D_1F(B),$$

and thus $H_0(D_1F \circ D_1G(X)) \approx H_0D_1F(A) \approx H_0D_1F \circ H_0D_1G(X)$, as desired.

## 6. Posets and the partition complex

In this section we describe the *partition complex* of a finite set together with some other related objects. These will play a crucial role in the argument.

6.1. **Posets and their nerves.** First we describe some general notation for simplicial sets associated to a poset.

Let $X$ be a poset. The simplicial nerve functor gives a fully faithful embedding of the category of posets into the category of simplicial sets. Thus, it is convenient to regard a poset as merely a certain kind of simplicial set, and so we use the same notation of a poset and its simplicial nerve; an *element* of a poset is exactly a *vertex* of the simplicial set. If $x_0 \le x_1 \le \cdots \le x_q$ is a finite increasing sequence of elements of $X$, we write $[x] = [x_0 \le x_1 \le \cdots \le x_q]$ for the corresponding $q$-simplex of the nerve. Non-degenerate simplices of the nerve correspond to chains in which each inequality is strict, which we notate as $[x_0 < x_1 < \cdots < x_q]$

We now suppose that the poset $X$ contains upper and lower bounds, which we denote $\underline{1}$ and $\underline{0}$ respectively. In what follows we will always assume $\underline{1} \neq \underline{0}$. We introduce the following notation:

(1) Let $\hat{X}$ denote the maximal subposet of $X$ which does not include $\underline{0}$; we also use this notation to denote the nerve of this poset.
(2) Let $\check{X}$ denote the maximal subposet of $X$ which does not include $\underline{1}$; we also use this notation to denote the nerve of this poset.
(3) Let $\ddot{X} = \hat{X} \cap \check{X}$ as posets; the nerve of $\ddot{X}$ is also an intersection of nerves.
(4) Let $X^\diamond$ denote the sub-simplicial set of $X$ defined by $\hat{X} \cup \check{X}$.
(5) Let $\overline{X}$ denote the pointed simplicial set defined by $X/X^\diamond$.

Note that neither $X^\diamond$ nor $\overline{X}$ are generally nerves of posets.

Observe that the simplicial sets $\hat{X}$ and $\check{X}$ are contractible (to the upper or lower bound) we have that $X^\diamond$ is weakly equivalent to the unreduced suspension of $\ddot{X}$, and $\overline{X}$ is weakly equivalent to the reduced suspension of $X^\diamond$.

The **order complex** of a poset $X$, as studied in algebraic combinatorics, is the simplicial complex whose $q$-simplices are strictly ordered chains $\overline{0} < x_0 < x_1 < \cdots < x_q < \overline{1}$ of elements in $X$ which are not upper or lower bounds. It corresponds precisely to the simplicial set $\ddot{X}$.

6.2. **Partition complex.** Fix $m \geq 0$. Let $P = P_m$ denote the poset of equivalence relations on the set $\underline{m} = \{1, \ldots, m\}$, ordered by refinement. Thus, a $q$-simplex of $P$ is a chain $[E_0 \leq \cdots \leq E_q]$ of equivalence relations, where we write $E \leq E'$ if $E$ is "finer" than $E'$, i.e., if $x \sim_E y$ implies $x \sim_{E'} y$ for all $x$ and $y$ in the set.

There is an evident action of the symmetric group $\Sigma_m$ on $P_m$, obtained from the action of $\Sigma_m$ on $\underline{m}$.

We obtain subcomplexes $\hat{P}_m$, $\check{P}_m$, $P_m^\diamond$ of $P_m$ and a quotient complex $\overline{P}_m = P_m/P_m^\diamond$, all of which inherit the $\Sigma_m$-action.

6.3. **Pure partitions.** Let $m = p^k$ for some $k \geq 0$. A **pure partition** of a set is an equivalence relation $E$ such that all the equivalence classes of $E$ have the same size. Thus, a partition $E$ of $\underline{m}$ is pure if and only if all the equivalence classes have order $p^j$, for some fixed $j \in \{0, \ldots, k\}$. We say that the **mesh** of such a pure partition is $j$; we write $\operatorname{mesh}(E) = j$.

The following is elementary but crucial.

6.4. **Proposition.** *Let $E$ be a partition of $\underline{m}$, and let $H \leq \Sigma_m$ be the subgroup consisting of all permutations which fix the partition $E$. Then $E$ is a pure partition if and only if $H$ acts transitively on $\underline{m} = \{1, \ldots, m\}$.*

6.5. **Corollary.** *If $H \leq \Sigma_m$ is a subgroup which acts transitively on $\underline{m} = \{1, \ldots, m\}$, then any $q$-simplex $[E_0 \leq \cdots \leq E_q]$ of $P_m$ which is fixed by $H$ consists of a chain of pure partitions.*

## 7. The relation between standard resolutions

This section contains the key observation of this paper: that the bar resolution of $\Delta$, which we need to understand in order to show that $\Delta$ is Koszul, is a linearization of the bar resolution of $\mathbb{T}$, which can be expressed in terms of partition complexes.

7.1. **Transitive $E$-homology.** Let $S \subset \underline{2}^{\underline{m}}$ denote the set of *surjective* functions $\underline{m} \to \underline{2}$, equipped with the evident $\Sigma_m$-action; it is a set of order $2^m - 2$.

For a spectrum $Z$ equipped with a $\Sigma_m$-action, define

$$Q(Z) \stackrel{\text{def}}{=} \mathrm{Cok}[E^\wedge_*(Z \wedge \Sigma^\infty_+ S)_{h\Sigma_m} \to E^\wedge_* Z_{h\Sigma_m}],$$

where the map is the one induced by projection $\pi\colon S \to *$. For lack of a better name, we will call this the **transitive $E$-homology** of the $\Sigma_m$-spectrum $Z$. We extend this notation to spaces: for a $\Sigma_m$-space $X$, we write $Q(X) := Q(\Sigma^\infty_+ X)$, while for a based $\Sigma_m$-space $Y$ we write $\widetilde{Q}(Y) := Q(\Sigma^\infty Y)$.

Note that the functor $Q$ is not a homology theory; however, it does take finite coproducts of $\Sigma_m$-spaces to direct sums. The name comes from the fact that $Q$ only sees $\Sigma_m$-orbits with transitive isotropy, according to the following proposition.

7.2. **Lemma.** *Suppose that $X$ is a finite $\Sigma_m$-set such that for all $x \in X$, the isotropy group $H \subseteq \Sigma_m$ of $x$ does not act transitively on $\underline{m}$. Then $Q(X) = 0$.*

*Proof.* Since $Q$ preserves finite coproducts, it suffices to consider $X = \Sigma_m/H$. If $H$ does not act transitively on $\underline{m}$, there exists a surjective function $f\colon \underline{m} \to \underline{2}$ which is invariant under the $H$-action, and thus $\sigma H \mapsto (\sigma H, f\sigma^{-1})\colon X \to X \times S$ is a $\Sigma_m$-equivariant section of the projection $X \times S \to X$. □

We will typically apply $Q$ and $\widetilde{Q}$ to *discrete* $\Sigma_m$-spaces, obtaining functors

$$Q\colon G\mathrm{Set} \to \mathrm{Mod}_{E_*}, \qquad \widetilde{Q}\colon G\mathrm{Set}_* \to \mathrm{Mod}_{E_*}.$$

Typically, we will have a $\Sigma_m$-equivariant simplicial set $X$, and we will apply $Q$ to each simplicial degree of $X$ separately; thus, if $X$ is a pointed $\Sigma_m$-equivariant simplicial set, $\widetilde{Q}(X)$ denotes the simplicial abelian group with

$$\widetilde{Q}(X)_q \stackrel{\text{def}}{=} \widetilde{Q}(X_q).$$

Because the functor $Q$ is actually defined on $\Sigma_m$-spectra, it is functorial not merely with respect to maps of $\Sigma_m$-sets, but also with respect to transfers. That is, we have the following observation.

7.3. **Proposition.** *The functor $Q$ is equipped with the structure of a $\Sigma_m$-Mackey functor.*

Furthermore, because $Q$ is really defined on $p$-local spectra, we have the following "transfer splitting" result.

7.4. **Corollary.** *Let $Z$ be a finite $\Sigma_m$-set with order prime to $p$. Then there exist maps*

$$Q(X) \xrightarrow{i} Q(X \times \Sigma_m/H) \xrightarrow{j} Q(X),$$

*natural in the $\Sigma_m$-set $X$, such that $ji$ is an isomorphism.*

*Proof.* If $Z$ is a finite $G$-set with order prime to $p$, it is standard that the composite

$$\Sigma^\infty_+(*)_{hG} \to \Sigma^\infty_+(Z)_{hG} \to \Sigma^\infty_+(*)_{hG}$$

of the transfer along $Z \to *$ followed by the projection along $Z \to *$ is a $p$-local equivalence. Applying $Q$ to this sequence gives the desired result. □

7.5. **Transitive $E$-homology as a linearization.** Fix a space $W$ equipped with a $\Sigma_m$-action. Let $F\colon h\text{Spectra} \to \text{Mod}_{E_*}$ be the functor defined by

$$F(Y) \overset{\text{def}}{=} E_*^{\wedge}(\Sigma_+^{\infty} W \wedge Y^{\wedge m})_{h\Sigma_m}.$$

Let $\mathcal{L}_F\colon h\text{Spectra} \to \text{Mod}_{E_*}$ be the linearization of $F$, as described in §5.

7.6. **Proposition.** *For all spaces $X$ there is a natural isomorphism*

$$\mathcal{L}_F(\Sigma_+^{\infty} X) \to Q(W \times X^m).$$

*Proof.* The sequence

$$Y^{\wedge m} \wedge \Sigma_+^{\infty} S \to (Y \vee Y)^{\wedge m} \xrightarrow{(p_1,p_2)} Y^{\wedge m} \vee Y^{\wedge m}$$

is a split cofibration sequence in the category of $\Sigma_m$-spectra, from which it follows that the map $\gamma_F\colon \perp F(Y) \to F(Y)$ is isomorphic to the map $E_*^{\wedge}(\Sigma_+^{\infty} W \wedge Y^{\wedge m} \wedge \Sigma_+^{\infty} S)_{h\Sigma_m} \to E_*^{\wedge}(\Sigma_+^{\infty} W \wedge Y^{\wedge m})_{h\Sigma_m}$ induced by projection $S \to *$. □

7.7. **Transitive $E$-homology of the partition complex.** We have the following.

7.8. **Proposition.** *Let $P = P_m$ be the nerve of the partition poset. Then the sequence of simplicial abelian groups*

$$0 \to Q(\ddot{P}) \xrightarrow{(incl.,-incl.)} Q(\hat{P}) \oplus Q(\check{P}) \xrightarrow{(incl.,incl.)} Q(P) \to \widetilde{Q}(\overline{P}) \to 0$$

*is exact.*

*Proof.* For $q \geq 0$, let $P_q$ denote the $q$-simplices of $P$. Then

$$P_q \approx (P_q - P_q^{\diamond}) \amalg (\hat{P}_q - \ddot{P}_q) \amalg (\check{P}_q - \ddot{P}_q) \amalg \ddot{P}_q$$

as $\Sigma_m$-sets, and the result follows because $Q$ preserves coproducts. □

7.9. **The partition complex and $E_\infty$-operads.** A **symmetric sequence** is a functor $A\colon \Sigma \to \text{Spaces}$, where, $\Sigma$ denotes the groupoid of finite sets and isomorphisms. A symmetric sequence $A$ determines a functor

$$\mathcal{C}_A\colon \text{Spaces} \to \text{Spaces} \quad \text{by} \quad \mathcal{C}_A(X) \overset{\text{def}}{=} \coprod_{m \geq 0} (A(\underline{m}) \times X^m)_{\Sigma_m},$$

and the assignment $A \mapsto \mathcal{C}_A$ is functorial. There is a monoidal product $A, B \mapsto A \circ B$, which has the property that $\mathcal{C}_{A\circ B} \approx \mathcal{C}_A \mathcal{C}_B$. This monoidal product satisfies the formula

$$(A \circ B)(S) \approx \coprod_{n,\, f\colon S \to \underline{n}} A(\underline{n}) \times \prod_{s \in S} B(f^{-1}(s))$$

where the coproduct runs over integers $n \geq 0$ and functions $f\colon S \to \underline{n}$. An operad is a monoid with respect to this monoidal product.

If $O$ is an operad, then $\mathcal{B}(O) = \mathcal{B}(O, O, O)$ is a simplicial object in symmetric sequences.

7.10. **Proposition.** *Let $O$ be the non-unital $E_\infty$-operad in spaces. For each $m \geq 0$ there is a map $\mathcal{B}(O)(\underline{m}) \to P_m$ of simplicial $\Sigma_m$-spaces, which is a weak equivalence of spaces in each simplicial degree. That is, $O^{\circ(q+2)}(\underline{m}) \to (P_m)_q$ is a weak equivalence.*

*Proof.* A standard and well-known combinatorial argument. In fact, taking $O$ to be the non-unital commutative operad, with $O(S)$ a one-point space for all non-empty $S$, gives an isomorphism $\mathcal{B}(O)(\underline{m}) \approx P_m$ of simplicial $\Sigma_m$-spaces. □

We may consider the monad $\widetilde{C} = C_O$ associated to $O$, together with its associated monadic bar construction $\mathcal{B}(\widetilde{C}) = \mathcal{B}(\widetilde{C}, \widetilde{C}, \widetilde{C})$. Evaluating at a space $X$ gives a simplicial space $\mathcal{B}(\widetilde{C})(X) = \mathcal{B}(\widetilde{C}, \widetilde{C}, \widetilde{C}(X))$, and applying (7.10) leads to the following.

7.11. **Proposition.** *There is an isomorphism*

$$\mathcal{B}(\widetilde{C})(X) \approx \coprod_{m\geq 0} (P_m \times X^m)_{h\Sigma_m},$$

*natural in CW-complexes $X$, in the category of simplicial objects in the homotopy category of spaces. In particular, for each $q \geq 0$ there are natural weak equivalences*

$$\widetilde{C}^{\circ(q+2)}(X) \approx \coprod_{m\geq 0} ((P_m)_q \times X^m)_{h\Sigma_m}.$$

Observe that there is a coproduct decomposition

$$\mathcal{B}(\widetilde{C}) \approx \coprod_{m\geq 1} \mathcal{B}(\widetilde{C})\langle m\rangle$$

in the category of simplicial functors, so that there are natural weak equivalences

$$\mathcal{B}(\widetilde{C})\langle m\rangle(X) \approx (P_m \times X^m)_{h\Sigma_m}.$$

7.12. **A fundamental observation.** Our approach to proving that $\Delta$ is a Koszul ring comes from the following observation: if we apply $E$-homology to $\mathcal{B}(\widetilde{C})(X)$ in each simplicial degree, and then "linearize" with respect to $X$, and set $X = *$, then what we obtain is the bar complex $\mathcal{B}(\Delta)$ for the ring $\Delta$. This linearization, in turn, turns out to be exactly the transitive homology of the partition complex.

7.13. **Proposition.** *For $m = p^k$, there is an isomorphism of simplicial abelian groups*

$$\mathcal{B}(\Delta)[k] \to Q(P_m).$$

*Furthermore, this isomorphism carries the subobjects $\hat{\mathcal{B}}(\Delta)[k]$, $\check{\mathcal{B}}(\Delta)[k]$, and $\ddot{\mathcal{B}}(\Delta)[k]$ isomorphically to the subobjects $Q(\hat{P}_m)$, $Q(\check{P}_m)$, and $Q(\dot{P}_m)$.*

Note that if $m$ is not a power of $p$, then $\widetilde{Q}(\overline{P}_m) = 0$. We give the proof of (7.13) below.

7.14. **Corollary.** *For $m = p^k$, $k \geq 0$, there is an isomorphism of simplicial abelian groups*

$$\overline{\mathcal{B}}(\Delta)[k] \approx \widetilde{Q}(\overline{P}_m).$$

*Proof.* Immediate using (7.13) to compare the exact sequences of (4.3) and (7.8). □

Now we relate the simplicial object $\mathcal{B}(\widetilde{C})(X)$ with the non-unital version $\widetilde{\mathbb{T}}$ of the algebraic approximation functor of §3.8.

7.15. **Proposition.** *If $X$ is a finite $\Sigma_m$-set, $m = p^k$, then there is an isomorphism*

$$\mathcal{B}(\widetilde{\mathbb{T}})\langle m\rangle(E_*X) \xrightarrow{\alpha} E_*^\wedge \mathcal{B}(\widetilde{C})\langle m\rangle(X)$$

*of simplicial $E_*$-modules.*

*Proof.* This amounts to the fact that

$$(\widetilde{\mathbb{T}}^{\circ q})\langle m\rangle(E_*X) \approx E_*^\wedge(O^{\circ q}(\underline{m}) \times X^m)_{h\Sigma_m},$$

where $O$ is a non-unital $E_\infty$-operad, using the fact that the algebraic approximation functor is an isomorphism on finitely generated free $E_*$-modules such as $E_*X$ (3.5)(3). □

*Proof of* (7.13)*.* By (7.15) and (7.11), there are isomorphisms

$$\mathcal{B}(\widetilde{\mathbb{T}})\langle m\rangle(E_*X) \approx E_*^\wedge \mathcal{B}(\widetilde{C})\langle m\rangle(X) \approx E_*^\wedge(P_m \times X^m)_{h\Sigma_m}$$

of simplicial $E_*$-modules, natural in $\Sigma_m$-sets $X$. Applying (7.6), we see that

$$\mathcal{L}_{\mathcal{B}(\widetilde{\mathbb{T}})\langle m\rangle}(\Sigma_+^\infty X) \approx Q(P_m \times X^m).$$

The isomorphism $\mathcal{B}(\Delta)[k] \approx Q(P_m)$ follows from the isomorphism

$$\mathcal{L}_{\mathcal{B}(\mathbb{T})}(M) \approx \mathcal{B}(\Delta) \otimes_{E_0} M$$

of (5.8).

To see that each of these isomorphisms induces isomorphisms of the relevant subobjects

$$\hat{\mathcal{B}}(\Delta)[k] \approx Q(\hat{P}_m), \quad \check{\mathcal{B}}(\Delta)[k] \approx Q(\check{P}_m), \quad \ddot{\mathcal{B}}(\Delta)[k] \approx Q(\ddot{P}_m),$$

we recall from (5.9) that the summands $\Delta[k_{q+1}] \otimes_{E_0} \cdots \otimes_{E_0} \Delta[k_0]$ of $\Delta^{\otimes(q+2)}$ come from the linearization of the pure part $\mathbb{T}\langle m_{q+1}\rangle \circ \cdots \circ \mathbb{T}\langle m_0\rangle$ of $\widetilde{\mathbb{T}}^{\circ(q+2)}$, where $m_i = p^{k_i}$.

On the other hand, since $Q$ vanishes on $\Sigma_m$-orbits whose isotropy does not act transitively on $\underline{m}$ (7.2), we see from (6.5) that $Q(P_m)$ depends only on the the pure part of the partition complex; i.e., $Q(P_{m,q}) = Q(P_{m,q}^{\text{pure}})$ where $P_{m,q}$ is the set of $q$-simplices of $P_m$ and $P_{m,q}^{\text{pure}} \subseteq P_{m,q}$ is the subset consisting of chains of pure partitions.

Thus $\mathcal{B}_q(\Delta)[k]$ is actually isomorphic to $Q$ of the subset of $P_{m,q}$ consisting of chains $[E_0 \leq \cdots \leq E_q]$ consisting of pure partitions with $\mathrm{mesh}(E_k) = m_0 + \cdots + m_k$. Tracing through the isomorphism, we see that the summand $\hat{\mathcal{B}}_q(\Delta)[k]$ corresponds to the subset of chains of pure partitions with $m_0 > 0$, while $\check{\mathcal{B}}_q(\Delta)[k]$ corresponds to the subset of chains of pure partitions with $m_q > 0$. The claim follows. □

## 8. Bredon homology of the partition complex

In this section we complete the proof that the ring $\Gamma$ of power operations for Morava $E$-theory is Koszul, by applying a result of Arone, Dwyer, and Lesh [ADL16], by proving that the simplicial abelian group $\widetilde{Q}(\overline{P}_m)$ has homology only in degree $m$.

8.1. *Remark.* The original version of this paper contained a extensive proof of this fact, which can be viewed as an elaborate generalization of the argument of Solomon and Tits [Sol69] on the homotopy type of the Tits building of a BN-pair, as realized in the case of the group $G = GL(k, \mathbb{F}_p)$ with its usual BN-structure. The author is rather fond of the original proof, but there is no reason to reproduce it here in light of the Arone-Dwyer-Lesh result. It can be found in the first (version 1) arXiv posting of this paper.

8.2. **Bredon homology of partition complexes.** We first identify a special case of the theorem of Arone-Dywer-Lesh, which we will apply to our calculation.

8.3. **Proposition** (Special case of [ADL16, Cor. 1.2])**.** *Let $m = p^k$ for some $k \geq 1$, and let $M$ be a Mackey functor for $G = \Sigma_m$ taking values in $\mathbb{Z}_{(p)}$-modules, which satisfies the following hypotheses.*

(i) *For all finite $G$-sets $X$ and $Z$ such that the order of $Z$ is prime to $p$, the composite*
$$M(X) \to M(X \times Z) \to M(X)$$
*is an isomorphism, where the maps in question are respectively the contravariant and covariant maps induced by the projection $X \times Z \to X$ of finite $G$-sets.*

(ii) *If $H \leq G$ is a subgroup which acts non-transitively on $\underline{m} = \{1, \dots, m\}$, then $M(G/H) \approx 0$.*

*Then there are isomorphisms*
$$\widetilde{H}_j^{\Sigma_m}(P_m^\diamond; M) \approx \begin{cases} 0 & \text{if } j \neq k-1, \\ M(\Sigma_m/V) \otimes_R \mathrm{St}_k & \text{if } j = k-1, \end{cases}$$
*where $V \leq \Sigma_m$ is a subgroup isomorphic to $(\mathbb{Z}/p)^k$, $R = \mathbb{Z}_{(p)}[\mathrm{Aut}(V)] = \mathbb{Z}_{(p)}[GL_k(\mathbb{F}_p)]$, and $\mathrm{St}_k$ is the Steinberg representation of $GL_k(\mathbb{F}_p)$.*

Recall the transitive $E$-homology functor $Q$ of the previous section. Restricted to finite $G$-sets, it defines a Mackey functor, which we also denote by $Q$.

8.4. **Proposition.** *For a based simplicial set $X$ with $G = \Sigma_m$ action, the homology of the simplicial abelian group $\widetilde{Q}(X)$ is precisely* (reduced) Bredon homology *of $X$ with coefficients in $Q$:*
$$H_* \widetilde{Q}(X) \approx \widetilde{H}_*^{\Sigma_m}(X; Q).$$

*Proof.* This is simply the definition of Bredon homology of a $G$-simplicial set. □

We obtain the result we need as a corollary of (8.3).

8.5. **Corollary.** *For $m = p^k$ and $k \geq 1$ we have that*
$$H_j \widetilde{Q}(X) = \widetilde{H}_j^{\Sigma_m}(\overline{P}_m; Q) = 0 \qquad \text{if } j \neq k.$$

*Proof.* As $\overline{P}_m = P_m/P_m^\diamond$ and $P_m$ is equivariant contractible, we see that $\overline{P}_m$ is equivalent to the reduced suspension of $P_m^\diamond$, so $H_j^{\Sigma_m}(\overline{P}_m; Q) = H_{j-1}^{\Sigma_m}(P_m^\diamond; Q)$. The claim then follows from (8.3) once we see that $Q$ satisfies (i) and (ii).

For condition (i), recall that $Q$ is defined as a quotient of the Mackey functor $X \mapsto E_*^\wedge(X_{\Sigma_m})$, which also satisfies property (i), since the composite $X_{h\Sigma_m} \to (X \times Z)_{h\Sigma_m} \to X_{h\Sigma_m}$ of transfer followed by projection is a $p$-local equivalence when the order of $Z$ is prime to $p$.

Condition (ii) is satisfied by $Q$ by construction. □

Next we derive our special case from the theorem stated in [ADL16].

*Proof of* (8.3). This is a special case of [ADL16, Cor. 1.2], whose conclusion is precisely that stated in (8.3), depending on hypotheses on a $Z_{(p)}$-Mackey functor $M$ stated as (1)–(3) of [ADL16, Cor. 1.2]. They prove that our condition (i) implies their condition (1); this is stated as [ADL16, Lemma 3.8]. Their (2) and (3) are conditions on the values of $M(\Sigma_m/D)$ where $D \leq \Sigma_m$ is an elementary abelian subgroup which acts freely and non-transitively on $\underline{m} = \{1, \dots, m\}$. These conditions (2) and (3) are trivially implied by our (ii), since that implies $M(\Sigma_m/D) = 0$. □

8.6. *Remark.* Kathryn Lesh has pointed out that the special case (8.3) really is the "easy case" of the machinery of [ADL16]. In particular, the proof of the special case we use can be extracted entirely from sections 1–6 and 10 of their paper (the proof of the main theorem being completed in section 10). The key point is the hypothesis (in [ADL16, Prop. 5.4]) that certain collections of subgroups are "homologically $\mu(G/D)$-ample", which is a vanishing condition on the homology of a certain space with $G$-action, with twisted coefficients $\mu(G/D)$.

In the case relevant to the proof of their theorem, $\mu(G/D) = M(G/D)$ with $D \leq G = \Sigma_m$ a non-transitive elementary $p$ group. Our condition (ii) ensures that $M(G/D) = 0$ and thus this hypothesis of their Prop. 5.4 is trivially satisfied. Sections 7–9 of their paper do the hard work of showing that their sharper conditions (2) and (3) given rise to the needed hypothesis; these aren't needed if we only need the special case we use.

8.7. **Proof of the main theorem.** Now we can give the proof of our main theorem.

*Proof of* (1.7). Recall that the graded ring $\Gamma$ is isomorphic to the graded ring $\Delta$ (3.20), so it suffices to show that $\Delta$ is Koszul. Recall (4.4) that $\Delta$ is Koszul if $H_q(\overline{\mathcal{B}}(\Delta)[m]) = 0$ for $q < m$. By (7.14) there is are isomorphisms of simplicial abelian groups $\overline{\mathcal{B}}(\Delta)[k] \approx \widetilde{Q}(\overline{P}_m)$. By (8.4) the homology of this simplicial abelian group is the Bredon homology $\widetilde{H}_*^{\Sigma_m}(\overline{P}_m; Q)$ with coefficients in transitive homology Mackey functor $Q$, and the desired vanishing is (8.5). □

Finally, we prove that the Koszul resolution (4.7) associated to $\Gamma$ (or $\Delta$) has length $n+1$, where $n$ is the height of the formal group associated to $E$.

8.8. **Proposition.** *For a height $n$-Morava $E$-theory, we have that $C[k] := H_k\overline{\mathcal{B}}(\Delta)[k] \approx H_k^{\Sigma_{p^k}}(\overline{P}_{p^k}; Q) \approx 0$, and thus any $\Gamma$-module $N$ which is flat (resp. projective) over $E_0$ admits a flat (resp. projective) $\Gamma$-module resolution of length $n+1$.*

*Proof.* Recall (4.6) that since $\Delta$ is Koszul, each $C[k]$ is finitely generated and projective as a left $E_0$-module (and thus is free since $E_0$ is a local ring). We have

$$\begin{aligned}
\sum_{k=0}^{\infty} \operatorname{rank} C[k] \cdot T^k &= \Big(\sum_{k=0}^{\infty} (-1)^k (\operatorname{rank} \Delta[k]) \cdot T^k\Big)^{-1} && \text{by (4.6)} \\
&= \Big(\sum_{k=0}^{\infty} \operatorname{rank} \Gamma[k] \cdot (-T)^k\Big)^{-1} && \text{using } \Gamma[k] \approx \Delta[k] \text{ (3.20)} \\
&= (1+T)(1+pT)\cdots(1+p^{n-1}T) && \text{by (3.21),}
\end{aligned}$$

whence $\operatorname{rank} C[k] = 0$ if $k > n$. □

8.9. *Remark.* This actually shows the ranks of the $C[k]$ are "Gaussian binomial coefficients":

$$\operatorname{rank} C[k] = |\{\text{subspaces } V \subset \mathbb{F}_p^n \text{ with } \dim V = k\}|.$$

## References


[And95] Matthew Ando, *Isogenies of formal group laws and power operations in the cohomology theories* $E_n$, Duke Math. J. **79** (1995), no. 2, 423–485.

[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland, *The sigma orientation is an* $H_\infty$ *map*, Amer. J. Math. **126** (2004), no. 2, 247–334.

[ADL16] G. Z. Arone, W. G. Dwyer, and K. Lesh, *Bredon homology of partition complexes*, Doc. Math. **21** (2016), 1227–1268.

[AM99] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788.

[BF15] Tobias Barthel and Martin Frankland, *Completed power operations for Morava E-theory*, Algebr. Geom. Topol. **15** (2015), no. 4, 2065–2131.

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, $H_\infty$ *ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986.

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200.

[JM04] B. Johnson and R. McCarthy, *Deriving calculus with cotriples*, Trans. Amer. Math. Soc. **356** (2004), no. 2, 757–803 (electronic).

[Kas95] Takuji Kashiwabara, $K(2)$*-homology of some infinite loop spaces*, Math. Z. **218** (1995), no. 4, 503–518.

[May70] J. Peter May, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Mathematics, Vol. 168, Springer, Berlin, 1970, pp. 153–231.

[McC83] James E. McClure, *Dyer-Lashof operations in K-theory*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 1, 67–72.

[PP05] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005.

[Pri70] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[Rez08] Charles Rezk, *Power operations for Morava E-theory of height* 2 *at the prime* 2 (2008), available at `arXiv:0812.1320(math.AT)`.

[Rez09] ———, *The congrugence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379, available at `arXiv:0902.2499`.

[Rez12] ———, *Modular isogeny complexes*, Algebraic & Geometric Topology **12** (2012), no. 3, 1373–1403, available at `:arXiv:1102.5022`.

[Rez13] ———, *Power operations in Morava E-theory: structure and calculations* (2013), available at `http://www.math.uiuc.edu/~rezk/power-ops-ht-2.pdf`.

[Sol69] Louis Solomon, *The Steinberg character of a finite group with BN-pair*, Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968), Benjamin, New York, 1969, pp. 213–221.

[Str97] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), no. 2, 161–208.

[Str98] N. P. Strickland, *Morava E-theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779, available at `arXiv:math/9801125`.

[Zhu14] Yifei Zhu, *The power operation structure on Morava E-theory of height 2 at the prime 3*, Algebr. Geom. Topol. **14** (2014), no. 2, 953–977.

[Zhu15] ———, *Modular equations for Lubin-Tate formal groups at chromatic level 2* (2015), available at `arXiv:1508.03358`.



Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL
*E-mail address*: `rezk@math.uiuc.edu`